19 Partial energy balances

"Wir lernen mit Augen des Geistes sehen, ohne die wir, wie überall, so besonders auch in der Naturforschung, blind umher tasten."

Johann Wolfgang Goethe: Vergleichende Anatomie (GA17:252).

PROBLEMS

Often the fundamental local momentum balance is much too detailed for the analysis of the motions, maybe the fundamental balance cannot be solved for given boundary conditions or the details of the solution are not of interest.

On the other hand the global balances of momentum and spin are often much too crude. For practical purposes, in particular for forcibly driven systems with exactly or approximately only few degrees of freedom, there is a need for descriptions 'between' the two 'extremes'.

MODELS

The model adopted is the class of simple forcibly driven bodies or systems of bodies of matter, the kinematics to be described by non-holonomic generalised speed and generalised position, the dynamics described by partial energy balances, balances of the aggregate momentum.

GOALS

The goal is to derive the state equations of aggregate mechanics of simple forcibly driven systems in a coherent fashion as weighted integrals of Cauchy's fundamental local momentum balance, weighted by partial velocities.

PLANS

Following the discussion of generalised kinematics, non-holonomic generalised speed in particular, generalised dynamics, the balance of aggregate momentum will be discussed, particularly in the explicit form of the Euler-Lagrangean equation.

As the simplest instances systems with holonomic generalised speed, the kinematics described by a degenerate state equation and the dynamics described by the Lagrangean equation will be introduced and the 'variational principles' of mechanics will be shown to be very special cases of only limited interest.

The exposition will conclude with remarks on some applications.

19.1 Generalised kinematics

PROBLEMS

MODELS

GOALS

PLANS

19.1.1 Elementary 'principles'

> "Die Mechanik ist ein Versuch, alle *wahren* Sätze, die wir zur Weltbeschreibung brauchen, nach einem Plane zu konstruieren."
>
> *Ludwig Wittgenstein: Tractatus logico-philosophicus, 6.343.*

19.1.1.1 AGGREGATION

The elementary 'principles' of classical mechanics,

- the balance of momentum, the integral of Cauchy's universal local balance of momentum,
- the balance of spin, the integral of the exterior vector product of Cauchy's universal local balance of momentum with the local distances from a reference point, and
- the balance of kinetic energy, the integral of the scalar product of Cauchy's universal local balance of momentum with the local velocities,

are all 'nothing but' weighted integrals, aggregations of the universal local balance of momentum of bodies of ponderable classical matter.

The pattern of their construction suggests how further 'principles' can be designed and constructed for any given purpose at hand without resorting to the variational considerations to be found in most derivations following Lagrange.

In line with the term 'local mechanics' for continuum mechanics the present chapter, as the foregoing, could have been called 'global mechanics' although the term 'aggregate' would be much more appropriate and suggestive. The traditional term 'generalised ' is being used only where 'aggregate' would be inappropriate, as in 'generalised speed' and 'generalised position'.

The terms 'variational mechanics' and 'analytical mechanics' are not being used. The changes of terminology are not intended to shy readers away but let them know what the author is talking about, as has been attempted throughout the whole treatise.

19.1.1.2 ENERGY BALANCES

Of general interest are the 'principle' of virtual energy or 'work' and the 'principle' of virtual power. In the present context the partial energy balances are of particular interest. They are the components of the balance of aggregate momentum for systems with non-holonomic generalised speed, of the Euler-Lagrangean equation.

The fundamental reference is still Lagrange's 'Méchanique analytique' and nearly all textbooks the author had a chance to inspect still follow the original exposition concerned with point masses and thus with holonomic generalised speed, although this theory does not even cover the motions of rigid bodies in the world around us.

Following the recommendations of Euler and D'Alembert in 1766 Friedrich II, the Great, appointed Lagrange to succeed Euler as director of the mathematical section, Direktor der Mathematischen Klasse, of the Prussian Academy of Sciences.

Lagrange's years in Berlin have been very productive (Wikipedia):

> "Lagrange stayed at Berlin for over twenty years, producing a large body of work and winning several prizes of the French Academy of Sciences. Lagrange's treatise on analytical mechanics (*Mécanique Analytique*, 4. ed., 2 vols. Paris: Gauthier-Villars et fils, 1888-89), written in Berlin and first published in 1788, offered the most comprehensive treatment of classical mechanics since Newton and formed a basis for the development of mathematical physics in the nineteenth century."

In 1787, the year after Frederick died, Lagrange moved to Paris at the invitation of Louis XVI. and in 1788-89 he published the '*Mécanique Analytique*'; he 'wrote' Boudri sloppily notes (2002/207).

Lagrange's basic ideas have been developed when he was still at Turin, largely influenced by correspondence with Euler. Lagrange himself starts every section of his work with extended notes on the historical development, on the 'state of the art' and the 'Translator's Introduction' provides a very knowledgeable abstract (1997/XI-XLI).

Ample explicit references and an extended discussion concerning the work of Lagrange, the concept of force in particular are to be found in Boudri's account (2002/207-227). As has been mentioned the exposition, trying to elucidate the role and the development of explicit and implicit metaphysics in the development of mechanics in the eighteenth century, suffers from Boudri's own inadequate metaphysics.

Incidentally Boudri's analysis of the role of meta-physics in the eighteenth century is not to be confused with the meta-physical 'solution' of the force problem proposed by Lopes Coelho, although the latter is also extensively referring to the discussions in the eighteenth century.

The problems at that time have been the motions of systems of rigid bodies. Routh's famous work still refers to this problem in its title 'Dynamics of a System of Rigid Bodies'. These and similar expositions exclude the most important parts of aggregate mechanics, irreversible dissipation and reversible storage of aggregate momentum, partial energies.

Morgenstern formally discusses weighted integrals of the universal local balance of momentum under the heading 'Der Satz von D'Alembert' at the end of the completely unsatisfactory 'Allgemeine Grundlegung' der 'Mechanik der Kontinua' (1961/69-83).

The 'basic principles' are introduced as 'axioms', as integrals of the local balance of momentum, which is only afterwards derived from the principles. The hopeless confusion, the circularity of the exposition is not be confused with the fact that all 'mythical' foundations, all initial problems are circular. To repeat: By their very nature, being axioms, they are provisional, stepping stones, roots in the swamp to help ourselves out of the morass.

19.1.1.3 EMMY NOETHER'S WORK

In the context of variational mechanics the construction of 'principles' or 'conservation' laws is connected with the name of Emmy Noether (Tobies, 2004). Lanczos shows that (1986/401):

> "Noether's theory can be conceived as an application of the theory of ignorable variables, although it is not usually treated from this viewpoint."

Ignorable variables, to be treated in connection with Jacobi's theory, have also been at the root of Hertz' mechanics.

"Noether considers variational problems having the property that the action integral remains invariant with respect to a group of transformations, applied either to the dependent or the independent variables. She shows that every parameter associated with such transformation leads to a corresponding conservation law. In fact it suffices to consider a group of infinitesimal transformations."

As examples Lanczos provides, although only for point masses, the derivation of the three 'conservation' laws which have been derived as elementary 'integrals' and providing the pattern of aggregations for the construction of others as may be required.

Noether's theorem, which 'had far-reaching repercussions in its applications to problems of contemporary field physics both in relativity and quantum theory', *will not further be referred to in the present treatise on classical mechanics for the simple reason that its proof rests on the principles to be derived and the limits of which will be explicitly discussed.*

The following quotation from Wikipedia may serve for ready reference (Baez, 2002):

"Noether's theorem is an amazing result which lets physicists get conserved quantities from symmetries of the laws of nature. Time translation symmetry gives conservation of energy; space translation symmetry gives conservation of momentum; rotation symmetry gives conservation of angular momentum, and so on.

This result, proved in 1915 by Emmy Noether shortly after she first arrived in Göttingen, was praised by Einstein as a piece of 'penetrating mathematical thinking'. It's now a standard workhorse in theoretical physics.

These days, students often first meet this theorem in a course on quantum field theory. That can make it seem more complicated than it really is. It works for classical field theory, not just quantum field theory. And it also works for the classical mechanics of a point particle! The proof looks a lot easier in this context - but it contains all the basic ideas which show up in the more fancy versions.

Of course the proof uses Lagrangians, but a proof can't help using the concepts which the theorem is about. In other words: if someone claims Noether's theorem says 'every symmetry gives a conserved quantity', they are telling a half-truth. *The theorem only applies to certain classes of theories. In its original version it applies to theories described by a Lagrangian, and the Lagrangian formalism does most of the work in proving the theorem.* There is also a version which applies to theories described by a Hamiltonian. Luckily, almost all the theories studied in physics are described by both a Lagrangian *and* a Hamiltonian." *Italics: MS.*

Indeed very lucky physicists, engineers are not so lucky: almost all the theories studied in engineering cannot be described by both a Lagrangian *and* a Hamiltonian.

As has been mentioned the qualification 'conservation' laws for balances already indicates that the theory is of interest only in the very special cases of conservative quantities. Classical mechanics is not at all restricted to such quantities, quite to the contrary. Momentum, spin and energy production, may be negative, not to forget diffusive flows, are dominating motions on macroscopic scale.

And last, but not least, it will be shown that the Lagrangean equation covers only the very special case of forcibly driven holonomic systems with holonomic generalised speeds, both restrictions useful in celestial mechanics, Lagrange's main subject of research at Berlin, but much too narrow for classical mechanics as it is practised today.

19.1.2 Generalised kinematics

19.1.2.1 Generalised position

The 'disadvantage' of the balance of kinetic energy is the extreme aggregation to only one scalar equation for the description of the global motions. This limitation can be relaxed, if for example only systems of 'partial' energy balances, the balance of aggregate momentum are being considered. For terminological convenience the term 'systems of' will be dropped, all the balances together will be considered as the balance of the aggregate momentum.

But before the dynamics can be treated the generalised or aggregate kinematics of bodies of matter needs to be described explicitly. If the configurations of a body of matter are constrained the configuration of the body is said to have only a finite number of degrees of freedom, and the body is said to move in some position space selected in accordance with the constraints adequate for the problem at hand.

Accordingly the configurations can be described in terms of the generalised position

$$\mathbf{s}_u = \mathbf{s}^f{}_u\,(t)\,, \qquad u = 1, \ldots, n\,.$$

In traditional jargon its components are called generalised 'coordinates', a name unnecessarily referring to the fact that a position is described by coordinates in the space of position chosen for convenience.

The name 'position' is preferred to the name 'location' with rigid body motions in mind, the concept of position including location and attitude. The space of positions is traditionally called configuration space.

The position space may be conceived as n-dimensional Euclidean space and thus the body of matter may be represented as a material point moving in that space, *to be constituted in due course*.

Further Cartesian reference frames are adopted for convenience, purposely avoiding the use of curvilinear coordinates introducing formal problems obscuring the essentials. The same approach has been taken in continuum mechanics.

19.1.2.2 GENERALISED SPEED

The motions of a constrained body of matter in the position space are said to have only a finite number of degrees of freedom of motion. The motions can thus be described in terms of the generalised speed

$$\mathbf{v}_v = \mathbf{v}^f{}_v (t), \qquad v = 1, \ldots, m ,$$

selected in accordance with the constraints adequate for the problem at hand. In traditional jargon its components are called generalised 'velocities'.

Generalised speeds and positions are conveniently represented by matrices, not 'vectors' in the ordinary sense, but generalised vectors, constituted by specific transformation rules. In that sense von Mises has used the term 'motors' in case of rigid body motions.

Traditionally the generalised position is denoted by q (Lanczos, 1986/6 ff; Sommerfeld, 1955/47), while Hertz uses p (1959/48 ff), which Sommerfeld would also have preferred. The deviation from the traditional notation, already suggested by Sommerfeld (1955/177 f), is felt to be warranted by the dramatic increase in transparency of the exposition.

19.1.2.3 NON-HOLONOMIC SYSTEMS

Due to degenerate 'forces' the components of the generalised speed may not be independent of each other. If the degrees of freedom in motion are less than those in configuration,

$$r = n - m > 0$$

the systems are non-holonomic, requiring additional kinematic equations

$$\mathbf{v}_w = \mathbf{C}_w (\mathbf{s}, \mathbf{v}), \qquad w = 1, \ldots, r .$$

describing the additional motion constraints. The classical example is that of a sharp edged wheel rolling on a plane (Sommerfeld, 1955/47 f, 236). Hertz in his 'Principles of Mechanics' has drawn attention to these conditions studied already earlier (Sommerfeld, 1955/47 f).

Kane[TR] restricts considerations to the class of 'simple' non-holonomic systems defined by the constitutive equation (1985/43):

$$\mathbf{v}_{w} = \mathbf{v}^{f}{}_{w\,0}(\mathbf{s}) + \mathbf{v}^{f}{}_{w\,u}(\mathbf{s})\, \mathbf{v}_{u}\,, \qquad w = 1, \ldots, r\,.$$

In rigid body motions and in potential flows, in forcibly moving systems, in zwangsgeführten Verbänden, in general, considered as examples further down, the constant term can be avoided by appropriate selection of the generalised speeds.

In the present treatise only holonomic systems will be considered. The treatment of rolling motions does not contribute to the understanding of the fundamentals. But the following case of non-holonomic speed is of great interest, theoretically and practically, permitting to develop a meta-theory among others of rigid body motions.

As an example of rolling motions Hamel treats the carriage with two wheels (1949/239). This carriage has been of interest as compass carriage in ancient China and is of considerable interest in differential geometry, permitting to identify the inner geometry of curved surfaces (Treitz, 2008).

19.1.2.4 NON-HOLONOMIC SPEED

In many cases of interest the generalised speed is non-holonomic, not integrable, and sometimes called 'quasi'-speed (Lurie, 2002/30-33; /391-395). This usage based on a much too narrow concept of speed is felt to be grossly misleading, completely obscuring the fundamental property concerned.

Hamel *expressis verbis* refers to the problem and does not use this term but only the term 'Quasikoordinaten' in inverted commata, quasi-positions (1949/476). The non-holonomic generalised speed is introduced in the last chapter 'On non-holonomic systems', at first only in an example (1949/467).

But after the derivation of the Euler-Lagrangean equation the section on holonomic systems starts with the remark (1949/481 f):

> "**236. Der Fall m = 0.** Es kann natürlich sein, daß … keine nichtholonomen Bedingungen vorliegen, dass man aber doch nichtholonome Geschwindigkeitsparameter einführt. Das ist z. B. beim starren Körper der Fall."

This short sentence finally refers to the fundamental distinction between non-holonomic systems and non-holonomic speeds.

The 'quasi'-velocities, the components of the non-holonomic generalised speed are not 'eingeführt', 'introduced', their very nature, being intrinsically non-integrable, cannot be 'transformed away'. What is being 'introduced' in the next section is the generalised position.

A direct analogue in thermodynamics comes to mind here. Heat supplied and work being done are non-integrable magnitudes. With the absolute temperature as 'integrating denominator' they can be integrated, the resulting concept being the entropy (Sommerfeld, 1965/21-31; Schmidt[Ern], 1953/77-82; Spalding, 1982/248-286).

The typical example of non-holonomic speed is the rotational speed of rigid bodies, which cannot be 'derived' from the angular position, it is not an 'angular' speed, despite the nearly 'universal' usage of this misleading, 'truly' *false* jargon, even by those who (should) know better but surprisingly do not care for an appropriate terminology (Kane[TR], 1985/15; Bottema, 1990/21; Lurie, 2002/30).

But as has been mentioned the usage of the terminology is not consistent in the literature and is definitely misleading in suggesting that the generalised position is not a state. Quite to the contrary, as the generalised position the generalised speed is a state of the system, dynamic state equations to be derived in the next sections.

19.1.2.5 MATHEMATICAL ARTIFICES

Constraints of generalised positions and speeds due to 'forces' are usually introduced as geometrical constraints (Föppl, 1910/75):

> "Wenn man eine Definition des Verbandes haben will, wird man zu sagen haben, daß darunter eine Vereinigung von Körpern oder Massen zu verstehen ist, zwischen denen geometrische Bedingungen bestehen, die zu Bewegungsbeschränkungen führen."

The geometrical constraints are mathematical artifices related to the limiting conditions of rigidity and/or others. In early theories the concept of Undurchdringlichkeit played a prominent role.

In the same spirit Föppl discusses the motion constraints due to stick/slip friction (1910/80):

> "Will man sich nun darauf beschränken, die Bewegung nur für solche Fälle zu untersuchen, bei denen tatsächlich kein Gleiten zu erwarten ist, so ist es bequem, *diese einschränkende Voraussetzung dadurch in den Ansatz eintreten zu lassen, daß man eine zwingende Bedingung , die von vornherein gar nicht vorhanden ist, fingiert und sie so, wie es geschehen ist, durch eine Differentialgleichung, der die virtuellen Koordinatenänderungen unterworfen werden sollen, zum Ausdruck bringt.*" *Italics: wide print in the reference.*

Mechanics abounds with 'degenerate' cases constituted by limiting conditions and the corresponding mathematical artifices, often the use of degenerate functions and related 'operations', each case causing an off-shoot, often a separate branch or rather twig of mechanics.

Although of utmost importance for technical applications the discussions of such special cases and the corresponding mathematical artifices are not subject of the present exercise, they do not contribute to the goal, they provide no insight into the fundamentals of mechanics.

The only cases to be treated later in detail as instances of the general theory are rigid bodies and incompressible ideal fluids due to their importance as basic models in many technical applications, often serving 'only' as models of axiomatic theories of their own.

If motion constraints and thus non-holonomic systems are excluded the equations of motion arrived at are 'stiff', with extremely wide ranges of time scales requiring special techniques for their integration. These are generalisations of the traditional methods.

Of particular interest are the Rosenbrock methods, the Bader methods, semi-implicit extrapolation methods, and Gear methods, backward differentiation methods (Press, 1992/727 ff). In the limit the equations are 'infinitely stiff', requiring mathematical artifices mentioned, typically motion constraints.

In the literature the concepts of virtual velocities and displacements play a dominant role while in the following derivations only the actual velocities need to be considered. As before virtual velocities will play a role in the context of stability investigations.

19.1.2.6 STATE EQUATIONS

The generalised speed or rather the aggregate momentum is subject to a dynamical state equation, to be discussed next. While in traditional expositions kinematics and dynamics of systems are intimately intertwined, in the present exposition the state equations are clearly separated as far as possible. Despite this formal separation they cannot be solved independently if, as usual, the aggregate inertia, momenta and forces depend on the generalised positions.

Further discussion of kinematics is not subject of this treatise. A rather technical discussion is to be found in Lesser's textbook, heavily leaning on Kane^{TR}'s theory and exposition, in the chapter on 'Configuration and Motion' (1985/77 ff). But this discussion is limited, as most other discussions of the subject, to 'mechanisms' consisting of rigid bodies (Wittenburg, 1977), including those without constraints, the classical example being the planetary system (Kane[TR], 1985/83):

> "The generalised coordinates determine the geometric configuration. This means that *any* point in the mechanism can be specified by giving the values of the generalised coordinates."

As the exposition has shown this view of generalised position or configuration, *alias* ' generalised coordinates', is much too narrow for practical applications in science and engineering.

19.1.3 Generalised position

19.1.3.1 KINEMATIC STATE EQUATION

The space of generalised position, for short the position space, will in general be different from the space of generalised speed and will be constituted by the kinematic state equation

$$d^{v}{}_{t}\, \mathbf{s}_{u} = \mathbf{K}_{u\,v}\,(\mathbf{s})\; \mathbf{v}_{v}\,.$$

often presented in the inverse form (Hertz, 1956/128; Lurie, 2002/393).

In view of the derivation of the Euler-Lagrangean equation Lurie correctly states (2002/393):

> " … *if* the quasi-velocities are introduced by means of the homogeneous linear form [*of the kinematic state equation*] and not be means of more general relation ships." [*Addition*]*: MS.*

The question arises here whether the generalised positions cannot in any case be 'introduced' or rather constituted by kinematic state equations in the format stated. The advantage of this kinematic state equation adopted is an extremely concise form of the dynamic state equation, the Euler-Lagrangean equation to be derived next. The simple systems introduced are even simpler than those considered by Kane[TR] and Lurie.

In the following the differential form of the kinematic state equation

$$\mathbf{K}^{-1}{}_{u\,v}\, d_{t}\, \mathbf{s}_{v} = \mathbf{v}_{u}$$

and its 'integral'

$$\int \mathbf{K}^{-1}{}_{u\,v}\, d\, \mathbf{s}_{v} = \int \mathbf{v}_{u}\, dt$$

will be of interest. Over a time increment dt the integrands are constant and the result is

$$\mathbf{K}^{-1}{}_{u\,v}\, \partial \mathbf{s}_{v} = \mathbf{v}_{u}\,.$$

In the special case of holonomic speed the kinematics 'degenerates'

$$\mathbf{K}_{u\,v} = \mathbf{U}_{u\,v}$$

and the kinematic state equation reduces to

$$d_t\, \mathbf{s}_u = \mathbf{v}_u\,.$$

Although the case of holonomic generalised speed, not even covering rigid body motions, is the exception rather than the rule it *figures prominently, often exclusively in most textbooks on theoretical mechanics* (Sommerfeld, 1955/177). Accordingly a special section will be devoted to this degenerate instance of the general theory and its ramifications, the 'variational principles' of mechanics.

19.1.3.2 NON-HOLONOMIC SYSTEMS

For non-holonomic *systems* the *additional equation of motion constraint* can be assumed to be of the linear type

$$\mathbf{v}_u = \mathbf{N}_{u\,v}(\mathbf{s})\, \mathbf{v}_v\,.$$

As has been stated this case will not be considered further.

While in the present exposition the generalised speed, the kinematic state of a body of matter is being considered as fundamental, Hertz introduces it under § 267. Problem 2 by the kinematic state equation (1956/128) in the section on motions of systems at the end of his First Book concluding with a cautious Note ending with the 'correct' remark (196/135):

> "But 'the correctness or incorrectness of these investigations can be neither confirmed nor contradicted by possible future experiences."

In view of the fact that very often the generalised speed is non-holonomic the standpoint taken here provides for a much more general basis for theoretical and practical applications.

Usually the generalised location is introduced with reference to the locations of material points in the observation space on which kinematic conditions are imposed (Lanczos, 1986/6 ff; Hertz, 1956/48-60; Kane^{TR}, 1985/37). Even in view of the examples to be considered in detail this practice is quite unacceptable.

In general the concept of location of material points is lacking operational interpretation and consequently it is expressly excluded from the present reconstruction of the fundamentals. Instead, the position will be introduced as state by the kinematic state equation.

19.1.3.3 LIMITED DEGREES OF FREEDOM

Very often material system in three-dimensional space are conceived as consisting of n material points and thus the position space has 'in principle' 3 n dimensions. But due to constraints the system has factually only 'much' less degrees of freedom

$$f \ll 3\,n\ .$$

It is restricted to move in a subspace, on a hyper-surface, the position space of only *f* dimensions.

While the latter statement concerning the subspace may remain unchanged the conception of material systems consisting of material points is not acceptable in the present treatise considering a material system as body or system of bodies of continuous distribution of ponderable matter. The original position space is thus conceived as Hilbert space, a space with Euclidean metric of infinite dimension.

The position spaces mentioned in the following are thus to be conceived as subspaces of the Hilbert space. An adequate discussion of their properties, curvature in particular, is far beyond the scope of the present treatise (Reinhardt, 1980/230-239).

The body described by the fields of mass density and of velocity in a region of three dimensional space is mapped onto the position space and, talking in terms of motions on hyper-surfaces, this is naïvely assumed to be curved by 'definition'. Accordingly the geometry assumed is Riemannian the metric determined by the mass distribution. All this has 'nothing' to do with our good old observation space but the position spaces of given material systems in that space.

19.1.4 Forcibly driven systems

In order to arrive at explicit results the kinematics has to be made explicit. As usual considerations will be limited to the class of 'simple', forcibly driven systems, zwangsgeführte Systeme, constituted by velocity fields subject to the constraint

$$v_i = \mathbf{v}_{iu}(\mathbf{s})\,\mathbf{v}_u\ ,$$

corresponding to the partial derivatives

$$\mathbf{v}_{iu} \equiv \boldsymbol{\partial}^{\mathrm{v}}{}_u\, v_i \equiv \partial\, v_i / \partial\, \mathbf{v}_u = \mathbf{v}_{iu}(\mathbf{s}, t)\ .$$

Following Kane[TR] the partial derivatives of the velocity field with respect to the generalised speed will for short be called 'partial velocities', though the term may be misleading (1985/40 ff). The simple systems introduced are even simpler than those considered by Kane[TR], and Lurie.

Thus, instead of dealing with the vector fields of local velocities only the generalised speed has to be dealt with. The partial velocities will be the influence or 'weight' fields for the construction of the partial energy balances, the balance of aggregate momentum.

In general neither the components of the generalised position nor the components of the generalised speed need to be all of the same dimension. In rigid body motions components of the generalised speed are typically translational and rotational speeds.

Only if all components of the generalised speed are of the same dimension as the translational velocity field, measured in the same units, the partial velocities are non-dimensional influence fields. In this case partial energies and powers, 'aggregate' momentum and forces, cannot be distinguished from 'integrated' momentum and forces, thus adding to the confusion.

The term generalised 'explicitly implies' the general case, in which aggregate magnitudes with components of different dimensions result. While conceptually this causes no problems numerical matrix algebra in general requires non-dimensional matrix elements.

These may be obtained by normalising or by 'non-dimensionalising', the latter preferably by consistent use of coherent systems of units, today typically the SI system of units. These pragmatic aspects are closely related to the fundamental aspects of invariance discussed.

The discussion so far has shown that the position spaces are certainly not geometrical spaces, but 'physical' spaces, constructed *ad hoc* for the representation of the motions of material systems in Euclidean observation space.

19.1.5 Generalised kinematics: applications

As usual the advantages of considering simple systems are directly useful results and theoretical insights at the same time. Among the results are clear-cut 'rules' for the simulation of motions and 'recipes' for the computational determination and the physical identification of the 'properties', the generalised tensors of phenomenological 'parameters'.

It is in this way that the author has used the aggregate momentum balance in all his professional work, mostly even only approximately, axiomatically in cases of non-simple systems and mostly even only qualitatively in designing strategies and in decision making.

In practice this description is often used as an approximation, the numbers of degrees of freedom in speed and position considered depending on the purpose and the degree of approximation, the precision required (Föppl, 1910/76).

The kinematic state equation, describing the kinematical relationship between the position and speed spaces selected, has to be set up explicitly in any particular case before the system can be simulated. The associated problems are not subject of the present treatise. As an example only rigid body kinematics will be treated explicitly for ready reference.

In addition to the kinematic state equation simulation of motions requires a dynamic state equation of the type

$$d^{v}{}_{t}\,\mathbf{v}_{u} = \mathbf{D}_{uv}(\mathbf{s}, \mathbf{v})$$

to be developed in the next section. For the following sub-section it is sufficient to introduce the mass distribution of the body of matter in terms of the density field and its weighted integral

$$\mathbf{I}^{v}{}_{uv} \equiv \int_{V} \boldsymbol{\nu}_{iu}\,\boldsymbol{\nu}_{iv}\,\rho\,dV = \int_{m} \boldsymbol{\nu}_{iu}\,\boldsymbol{\nu}_{iv}\,dm\ ,$$

the generalised tensor of aggregate inertia.

19.1.6 Transformations

19.1.6.1 Kinetic energy: invariant

In summary; a mechanical system may be considered as a single material point moving in n-dimensional Euclidean position space, its mass being the invariant total mass of the body of matter under consideration and its state of motion in the position space may be described in terms of generalised location and speed.

The adequate choice of the reference frames in the speed and position spaces may dramatically simplify the conceptual and the computational effort, although the latter is in most cases no longer a matter of concern.

In order to provide for objective descriptions of the motions the kinetic energy of the system in the observation space at the point in time considered has to be the same in all 'frames'. Accordingly the most direct and simple rule of transformation between generalised speed 'frames' is

$$E = \mathbf{I}^{v}{}_{uv}\,\mathbf{v}_{v}\,\mathbf{v}_{u}/2 = \text{invar}\ ,$$

where the generalised tensor **I** denotes the aggregate inertia, a measure of mass distribution at the given location in the frame adopted. Correspondingly the rule of transformation between 'frames' is

$$E = \mathbf{I}^{s}{}_{uv}\,d_{t}\,\mathbf{s}_{v}\,d_{t}\,\mathbf{s}_{u}/2 = \text{invar}\ .$$

These relationships directly provide for the transformation of the inertiae when changing from one generalised 'frame' to another.

For theoretical considerations these conditions of invariance can be phrased

$$\mathbf{v}^2 = \mathbf{I}^{\mathbf{v}}{}_{u\,v} / m\; \mathbf{v}_v\, \mathbf{v}_u = \text{invar}\,,$$

$$\mathbf{v}^2 = \mathbf{I}^{\mathbf{s}}{}_{u\,v} / m\; d_t\, \mathbf{s}_v\, d_t\, \mathbf{s}_u = \text{invar}\,,$$

respectively, in terms of the *invariant energy speed*

$$\mathbf{v}^2 \equiv (2\,E / m)$$

of the system along the path in the observation space at time under consideration.

19.1.6.2 METRICS: INTRODUCED

In terms of Riemannian geometry the mass specific aggregate moments of the mass distribution

$$\boldsymbol{\mu}^{\mathbf{v}}{}_{u\,v} \equiv \mathbf{I}^{\mathbf{v}}{}_{u\,v} / m\,,$$

$$\boldsymbol{\mu}^{\mathbf{s}}{}_{u\,v} \equiv \mathbf{I}^{\mathbf{s}}{}_{u\,v} / m\,,$$

represent the metrics of the generalised non-Cartesian 'frames' of generalised speed and position, respectively. The frames and their metrics are not only non-Cartesian in the ordinary sense but have physical dimensions other than that of lengths.

In general the aggregate inertiae and thus the 'physical' or 'dynamical' metrics are functions of the position of the system in the position space. Only for a completely rigid system and a speed space fixed in the system the metric of the latter is independent of the position. Rigid bodies in ideal fluids will provide an example.

Invariant are the motions of the material system in the observation space and the motions of the material point in the position space, the phenomena to be described. The metrics are those of the speed and position frames selected, neither of the observation space nor of the position space.

In theoretical mechanics the time element dt is introduced as further invariant and the invariance conditions may be phrased in terms of the 'line element' (Lanczos, 1986/17-24)

$$d\mathbf{s}^2 = \boldsymbol{\mu}^{\mathbf{v}}{}_{u\,v}\, \mathbf{v}_u\, \mathbf{v}_v\, dt^2 = \text{invar}\,,$$

$$d\mathbf{s}^2 = \boldsymbol{\mu}^{\mathbf{s}}{}_{u\,v}\, d\mathbf{s}_u\, d\mathbf{s}_v = \text{invar}\,,$$

respectively. Accordingly the *'line element'* is the distance of two neighbouring points *of the system* in the position space, and *not* two neighbouring points *of the position space* and should thus more adequately be called *'path element'*. If the mass of the system is not known or not defined the formulations in terms of energy speed, the speed along the path, and the path element are not practically 'useful'.

19.1.6.3 METRICS: TRANSFORMED

From any of the two invariance conditions the kinematic state equation

$$\mathrm{d}_t \mathbf{s}_u = \mathbf{K}_{uv}(\mathbf{s})\, \mathbf{v}_v ,$$

may be derived. Accordingly the following relationships hold between the inertiae

$$\mathbf{I}^{\mathbf{v}}{}_{uv} = \mathbf{I}^{\mathbf{s}}{}_{rs}\, \mathbf{K}_{ru}\, \mathbf{K}_{sv}$$

and the metrics

$$\boldsymbol{\mu}^{\mathbf{v}}{}_{uv} = \boldsymbol{\mu}^{\mathbf{s}}{}_{rs}\, \mathbf{K}_{ru}\, \mathbf{K}_{sv}$$

in the speed and position 'frames', respectively.

In textbooks on theoretical mechanics hardly any of these fundamentals are being adequately discussed. Usually the case of 'degenerate' kinematics

$$\mathrm{d}_t \mathbf{s}_u = \mathbf{v}_u ,$$

corresponding to the kinematic 'relation'

$$\mathbf{K}_{vu}(\mathbf{s}) = \mathbf{U}_{uv} ,$$

the unit matrix, traditionally denoted by the Kronecker delta

$$\mathbf{U}_{uv} \equiv \boldsymbol{\delta}_{uv} ,$$

is being introduced without any discussion.

At this stage the question concerning the metric of the position space can be answered. For degenerate kinematics both metrics introduced are the same

$$\boldsymbol{\mu}^{\mathbf{s}}{}_{uv} = \boldsymbol{\mu}^{\mathbf{v}}{}_{uv} .$$

and according to the definition of the speed metric

$$\boldsymbol{\mu}^{\mathbf{v}}{}_{uv} \equiv \int_m \mathbf{v}_{iu}\, \mathbf{v}_{iv}\, \mathrm{d}m / m$$

with the mass of the body of matter

$$m = \int_V \rho\, \mathrm{d}V$$

they reduce to the unit matrix, *the Cartesian metric if the partial speeds are orthogonal everywhere in the observation space*

$$\mathbf{\nu}_{i\,u}\,\mathbf{\nu}_{i\,v} = \mathbf{U}_{u\,v}\,.$$

Starting from any position frame a 'Cartesian' frame may be constructed and *vice versa* by local transformation

$$\mathrm{d}\mathbf{s}^{C}{}_{u} = \mathbf{A}_{u\,v}\,\mathrm{d}\mathbf{s}_{v}\,,$$

the transformation matrix to be determined from the invariance condition

$$\mathbf{A}^{2}{}_{u\,v} = \mathbf{A}_{u\,w}\,\mathbf{A}_{w\,v} = \boldsymbol{\mu}^{s}{}_{u\,v}\,.$$

EVALUATIONS/ASSESSMENTS

CONCLUSIONS

19.2 Aggregate dynamics

" 'On the contrary' said Lady Muriel, 'it is a specia1 delight to me to have a question discussed in this way – analysed and arranged so that one can understand it. Some books, that profess to argue out a question, are to me intolerably wearisome, simply because the ideas are all arranged. haphazard – a sort of *first come, first served*.'

'You are very encouraging,' Arthur replied, with a pleased look."

Lewis Carroll: Sylvie and Bruno Concluded (1988/532).

PROBLEMS

MODELS

GOALS

PLANS

19.2.1 Modes of aggregation

"The author's intention has been to escape from the conventional categories and standards of difficulty."

John Formby: An Introduction to the Mathematical Formulation of Self-Organising Systems; from the jacket text (1965).

The generalised motion is subject to the dynamical state equation, the equation of motion, to be formulated in each particular case. In simple cases this can be done on the basis of the principles discussed so far (Föppl, 1910/76-77):

> "Diese Hilfsmittel reichen auch in erheblich verwickelteren Fällen in der Regel vollständig aus, wenn man sie geschickt anzuwenden versteht. Aber der glückliche Gedanke, der nötig ist, um damit zum Ziele zu kommen, stellt sich manchmal nicht zur rechten Zeit ein. Es ist daher von Vorteil, wenn man allgemeine Methoden gefunden hat, nach denen die Bewegungsgleichungen stets gebildet werden können, ohne daß dabei ein besonderer erfinderischer Gedanke für die Behandlung jedes einzelnen Falles erforderlich wäre. Die wichtigste dieser Methoden ist die von Lagrange angegebene, und von ihr wird daher in diesem Abschnitt vorwiegend die Rede sein."

In the present, motion oriented treatise the classical derivation of the Lagrangean equations is not adequate, instead the method of Kane^{TR} is generalised, offering dramatic advantages (Kane^{TR}, 1985/X):

> " ... it is to say that it is unlikely that a way will be found to reduce formulating equations of motion for complex systems to a truly simple task, there does exist a method that is superior to to the classical ones in that its use leads to major savings in labor, as well as to simpler equations. Moreover, being highly systematic, this method is easy to teach. Focussing on motions, rather than configurations, it affords the analyst maximum physical insight. Not involving variations, such as those encountered in connection with virtual work, it can be presented on a relatively elementary mathematical level. Furthermore, it enables one to deal directly with non-

holonomic systems without having to introduce and subsequently eliminate Lagrange multipliers."

Complementing the generalised description of kinematics an aggregate description of momentum is introduced. The two sensible ways of aggregation are:

- the aggregate momentum balances, the integration of the universal local momentum balance multiplied by the local partial velocities

$$\int_V \partial^{\mathbf{v}}{}_u\, v_i\, [\rho\, d_t\, v_i = \partial_j\, \sigma_{j\,i} + \rho\, f_i]\, dV$$
$$= \int_V \mathbf{v}_{i\,u}\, [\rho\, d_t\, v_i = \partial_j\, \sigma_{j\,i} + \rho\, f_i]\, dV\,.$$

- the partial energy balances, the integration of the partial derivative of the local energy balance,

$$\partial^{\mathbf{v}}{}_u \int_V v_i\, [\rho\, d_t\, v_i = + \partial_j\, \sigma_{j\,i} + \rho\, f_i\, v_i]\, dV$$
$$= \partial^{\mathbf{v}}{}_u\, [d_t\, E = E^M + E^P]\,,$$

The short hand notation

$$\partial^{\mathbf{v}}{}_u\, .\ \equiv \partial\ . / \partial\, \mathbf{v}_u$$

for partial derivatives with respect to the generalised speed has been defined already earlier. Further down the corresponding notion for partial derivatives with respect to the generalised positions

$$\partial^{\mathbf{s}}{}_u\, . \equiv \partial\ . / \partial\, \mathbf{s}_u$$

and for time derivatives with respect to the generalised speed and position

$$d^{\mathbf{v}}{}_t\, .\ \equiv d\ . / d\, \mathbf{v}_u$$

and

$$d^{\mathbf{s}}{}_t\, .\ \equiv d\ . / d\, \mathbf{s}_u\,,$$

respectively, will be used as well.

In the following the usage of the notation for the time derivative in the speed space is not quite uniform but depending on the context. So the following versions will be found

$$d^{\mathbf{v}}{}_t\, . = \partial^{\mathbf{v}}{}_t\, . = \partial_t\, . = d_t\, .\,,$$

the latter two after the Euler-Lagrangean equation has been derived and it is understood that the reference frames are fixed in the speed space.

The present approach is closely related to Morgenstern's and Szabó's goal (1961/71 ff):

" … um der verbreiteten Betrachtung an 'kleinen Elementen', die oft so kunstvoll sind, daß sie nur der Kenner verstehen kann, weiterhin aus dem Wege zu gehen."

19.2.2 Partial energy balances: *en detail*

19.2.2.1 CONCEPTS DEFINED

The first aggregation

$$\int_m \boldsymbol{\nu}_{iu}\, d_t v_i\, dm = \int_V \boldsymbol{\nu}_{iu}\, \partial_j \sigma_{ji}\, dV + \int_m \boldsymbol{\nu}_{iu}\, f_i\, dm$$

and further

$$d_t \int_m \boldsymbol{\nu}_{iu}\, v_i\, dm - \int_m (d_t \boldsymbol{\nu}_{iu})\, v_i\, dm$$
$$= \int_V \partial_j (\boldsymbol{\nu}_{iu}\, \sigma_{ji})\, dV - \int_V (\partial_j \boldsymbol{\nu}_{iu})\, \sigma_{ji}\, dV$$
$$+ \int_m \boldsymbol{\nu}_{iu}\, f_i\, dm\,.$$

directly provides for the concepts and their interpretations.

For short this weighted integral of Cauchy's universal local balance of momentum of ponderable matter is called the balance of aggregate momentum

$$d_t \mathbf{M}_u - \mathbf{M}^C{}_u = \mathbf{M}^M{}_u + \mathbf{M}^T{}_u + \mathbf{M}^P{}_u\,,$$

with the aggregate momentum

$$\mathbf{M}_u \equiv \int_m \boldsymbol{\nu}_{iu}\, v_i\, dm = \int_m (\boldsymbol{\partial}^v{}_u\, v_i)\, v_i\, dm \equiv \mathbf{I}_{uv}\, \mathbf{v}_v\,,$$
$$= \boldsymbol{\partial}^v{}_u \int_m v_i\, v_i/2\, dm = \boldsymbol{\partial}^v{}_u\, E\,.$$

For the simple systems considered the aggregate momentum becomes

$$\mathbf{M}_u = \int_m \boldsymbol{\nu}_{iu}\, \boldsymbol{\nu}_{iv}\, dm\, \mathbf{v}_v \equiv \mathbf{I}_{uv}\, \mathbf{v}_v$$

with the aggregate inertia

$$\mathbf{I}_{uv} \equiv \int_m \boldsymbol{\nu}_{iu}\, \boldsymbol{\nu}_{iv}\, dm = \mathbf{I}_{uv}(\mathbf{s})$$

depending on the generalised position in the position space only.

The remaining 'components' of the balance are

- the convective momentum flow inside the body of matter due to the kinematics

  $$\mathbf{M}^C{}_u \equiv \int_m d_t \boldsymbol{\nu}_{iu}\, v_i\, dm\,,$$

- and the aggregate 'forces', that are partial powers (!), in detail:

- the diffusive momentum flow into the body of matter due to the stresses at the surface of the body of matter

$$\mathbf{M}^{M}{}_{u} \equiv \boldsymbol{\partial}^{\mathbf{v}}{}_{u}\, E^{M} \equiv \boldsymbol{\partial}^{\mathbf{v}}{}_{u} \int_{A} v_{i}\, \sigma_{ij}\, dA_{j} = \int_{A} \boldsymbol{\nu}_{iu}\, \sigma_{ij}\, dA_{j}$$

and the aggregate momentum productions in the system:

- the aggregate reversible storage and irreversible diffusion of momentum, due to the local diffusion and deformation of momentum

$$\mathbf{M}^{T}{}_{u} \equiv \boldsymbol{\partial}^{\mathbf{v}}{}_{u}\, E^{PM} \equiv -\, \boldsymbol{\partial}^{\mathbf{v}}{}_{u} \int_{V} \sigma_{ij}\, \partial_{j}\, v_{i}\, dV$$

$$= -\int_{V} \sigma_{ji}\, \partial_{i}\, \boldsymbol{\nu}_{ju}\, dV\ ,$$

- and the aggregate weight due to the force field, the intensity of the momentum production in the observation space

$$\mathbf{M}^{P}{}_{u} \equiv \boldsymbol{\partial}^{\mathbf{v}}{}_{u}\, E^{P} \equiv \boldsymbol{\partial}^{\mathbf{v}}{}_{u} \int_{m} v_{i}\, f_{i}\, dm$$

$$= \int_{m} \boldsymbol{\nu}_{iu}\, f_{i}\, dm\ .$$

respectively. Except for the aggregate 'force' due to diffusive momentum flow all other generalised 'forces' are defined as integrals over volume of the body or its mass distribution.

19.2.2.2 Terminology etc

The balance of aggregate momentum for simple forcibly driven holonomic systems with non-holonomic generalised speeds is perfectly suitable for treating a very wide class of problems. But most systems in continuum dynamics are not simple, are not 'zwangsläufig', their velocity fields can be linearly super-imposed only in case of potential flows.

Instead of 'generalised' the more intuitive terminology 'aggregate' suggested by the above integrals has been used consistently. In the literature the term 'reduced' is also to be found, reduced to the generalised speed, not to the generalised position. In German engineering jargon the aggregate inertia is thus called 'reduzierte Masse' (Sass, 1955/248-252).

Similarly the aggregate or 'reduced' inertia of the fluid surrounding a moving body, the 'hydrodynamic' inertia of translation is called 'added mass' although it is of tensorial character (Newman[JN], 1977/139-149). In the following it will be shown how this terminology comes about and how it can be generalised.

Traditionally the aggregate 'forces' are grouped differently

$$\mathbf{M}^{M}{}_{u} + \mathbf{M}^{P}{}_{u} = \mathbf{F}^{impr}{}_{u} + \mathbf{F}^{deform}{}_{u}$$

those due to 'impressed' surface and body 'forces', gravity forces in particular

$$\mathbf{F}^{impr}{}_{u} = \int_A \boldsymbol{\nu}_{i\,u}\, \sigma_{i\,j}\, dA_j + \int_V \rho\, f_i\, \boldsymbol{\nu}_{i\,u}\, dV$$

and those due to deformations of the partial velocity fields

$$\mathbf{F}^{deform}{}_{u} = -\int_V \sigma_{i\,j}\, \partial_j\, \boldsymbol{\nu}_{i\,u}\, dV\ .$$

In view of the endless discussion on the concept of force it is repeated that the concepts of force in elementary, Newtonian and Eulerian mechanics and in aggregate, Lagrangean mechanics are essentially different, wesensverschieden.

Aggregate 'forces' are no longer simply momentum diffusion into the body of matter and momentum production in the body of matter but partial powers of the latter and, in addition, reversible storage and irreversible dissipation of aggregate momentum due to deformation and momentum diffusion in the body. Referring to Boudri's exposition this change of usage is not a change in metaphysics but a change of subject and the corresponding change of the model adopted.

In a more formal jargon: as many other terms the term 'force' is used generically, its 'meaning' depends on the context. In the vast literature on the concept of force inspected this 'simple' observation has occasionally been found if the concept was used in very wide sense, but never if used in mechanics.

19.2.3 Partial energy balances: *en gross*

> "Die vollendete Form der Wissenschaft muß poetisch sein. Jeder Satz muß einen selbständigen Charakter haben – ein selbstverständliches Individuum, Hülle eines witzigen Einfalls sein."
>
> *Novalis: Fragmente und Studien 1797-1798 (1981/377).*

19.2.3.1 PARTIAL ENERGY BALANCES: *IMPLICIT*

Most elegantly the partial energy balances are obtained by the second aggregation

$$\partial^{v}{}_{u} \int_V v_i\, [\rho\, d_t\, v_i = \partial_j\, \sigma_{j\,i} + \rho\, f_i]\, dV\ ,$$

of Cauchy's universal balance of local momentum mentioned, which reduces to the partial derivative of the global balance of kinetic energy

$$\partial^{v}{}_{u}\,[d_t\,E = E^{M} + E^{P}]\ .$$

The partial derivative, with respect to the generalised non-holonomic speed, of the rate of change of the kinetic energy in the position space is arrived at by the product rule

$$\partial^{v}{}_{u}\,(d_t\,E) = d_t\,(\partial^{v}{}_{u}\,E) - (d_t\,\partial^{v}{}_{u})\,E\ ,$$

corresponding to the equation

$$d^{v}{}_{t}\,\mathbf{M}_{u} = d_t\,M_u - d^{s}{}_{t}\,\mathbf{M}_{u}$$

in terms of the aggregate momentum

$$\mathbf{M}_{u} = \partial^{v}{}_{u}\,E\ .$$

Introducing the kinematic state equation for simple systems in its inverse form

$$\mathbf{v}_{u} = \mathbf{K}^{-1}{}_{u\,v}\,d\,\mathbf{s}_{v}/\,dt$$

into the differential operator

$$d_t\,\partial^{v}{}_{u} = d_t\,...\,\partial\,...\,/\,\partial\,\mathbf{v}_{u} = d_t\,...\,\partial\,...\,/\,(\partial\,\mathbf{K}^{-1}{}_{u\,v}\,d_t\,\mathbf{s}_{v})\ .$$

Using the relationship derived earlier the operator becomes

$$d_t\,\partial^{v}{}_{u} = K_{v\,u}\,\partial\,...\,/\,\partial\,\mathbf{s}_{v} \equiv K_{v\,u}\,\partial^{s}{}_{v}... \equiv \partial^{s\,v}{}_{u}...$$

The same result is obtained based on the identity

$$(d_t\,\partial^{v}{}_{u})\,E \equiv d^{s}{}_{t}\,\mathbf{M}_{u} = K_{v\,u}\,\partial^{s}{}_{v}\,... \equiv \partial^{s\,v}{}_{u}...$$

and thus the balance of aggregate momentum, the generalised Lagrangean equation

$$d^{v}{}_{t}\,\mathbf{M}_{u} = d_t\,\partial^{v}{}_{u}\,E - \partial^{s\,v}{}_{u}\,E = \mathbf{M}^{M}{}_{u} + \mathbf{M}^{T}{}_{u} + \mathbf{M}^{P}{}_{u}\ .$$

19.2.3.2 PARTIAL ENERGY BALANCE: *EXPLICIT*

As it stands the implicit form of the partial energy balances, the generalised Lagrangean equation is not particularly useful, but in its explicit form of the Euler-Lagrangean equation it is extremely useful. It will not only provide important theoretical insights but permit ready applications to realistic problems, starting with the motions of rigid bodies usually treated inadequately.

The partial derivative, with respect to generalised speed, of the rate of change of the kinetic energy in the position space

$$d^{v}{}_{t}\,\mathbf{M}_{u} = d_t\,\mathbf{M}_{u} - d^{s}{}_{t}\,\mathbf{M}_{u}\ ,$$

corresponding to the equation in terms of momentum

$$\partial^{\mathbf{v}}{}_{u}\,(d_t\,E) = d_t\,(\partial^{\mathbf{v}}{}_{u}\,E) - (d_t\,\partial^{\mathbf{v}}{}_{u})\,E\,,$$

can explicitly be arrived at if the kinematic state equation and the following relationships for the kinetic energy and its derivatives are observed:

the kinetic energy

$$E = \mathbf{I}_{u\,v}\,\mathbf{v}_u\,\mathbf{v}_v/2$$

the aggregate momentum

$$\mathbf{M}_u \equiv \partial^{\mathbf{v}}{}_{u}\,E \equiv \mathbf{I}_{u\,v}\,\mathbf{v}_v$$

with the symmetric aggregate inertia

$$\mathbf{I}_{u\,v} \equiv \partial^{\mathbf{v}}{}_{u}\,\mathbf{M}_v \equiv \partial^{\mathbf{v}}{}_{u}\,\partial^{\mathbf{v}}{}_{v}\,E$$

solely depending on the generalised location

$$\mathbf{I}_{u\,v}\,(\mathbf{s}) = \mathbf{I}_{v\,u}\,(\mathbf{s})$$

and thus

$$\partial^{\mathbf{v}}{}_{u}\,\mathbf{I}_{u\,v} = \mathbf{0}_v\,.$$

19.2.3.3 TOTAL RATE OF CHANGE

The total rate of change of momentum in the position space is the sum of the rate of change in the speed space and the additional rate of changes due the relative 'motion' of the speed space in the position space

$$\begin{aligned} d_t\,\mathbf{M}_u &= \partial_t\,\mathbf{M}_u + d_t\,\mathbf{s}_w\,\partial^{\mathbf{s}}{}_{w}\,\mathbf{M}_u \\ &= \partial_t\,\mathbf{M}_u + \mathbf{v}_v\,\mathbf{K}_{w\,v}\,\partial^{\mathbf{s}}{}_{w}\,\mathbf{M}_u \\ &= \partial_t\,\mathbf{M}_u + \mathbf{v}_v\,\partial^{\mathbf{s}\,\mathbf{v}}{}_{v}\,\mathbf{M}_u\,. \end{aligned}$$

The second term represents the convective transport in the narrow sense.

The total rate of change in question becomes

$$d_t\,\mathbf{M}_u = \partial_t\,\mathbf{M}_u + \mathbf{v}_v\,\partial^{\mathbf{s}\,\mathbf{v}}{}_{v}\,(\mathbf{I}_{u\,w}\,\mathbf{v}_w) - \partial^{\mathbf{s}\,\mathbf{v}}{}_{u}\,(\mathbf{I}_{v\,w}\,\mathbf{v}_v\,\mathbf{v}_w/2)\,,$$

that is explicitly

$$\begin{aligned} d_t\,\mathbf{M}_u &= \partial_t\,\mathbf{M}_u - \mathbf{M}^{C}{}_{u} \\ &= \partial_t\,\mathbf{M}_u + \mathbf{v}_v\,(\partial^{\mathbf{s}\,\mathbf{v}}{}_{v}\,\mathbf{M}_u - \partial^{\mathbf{s}\,\mathbf{v}}{}_{u}\,\mathbf{M}_v). \end{aligned}$$

The format of the resulting generalised 'gyroscopic force' arrived at

$$\mathbf{M}^{C}{}_{u} = -\,\mathbf{v}_v\,(\partial^{\mathbf{s}\,\mathbf{v}}{}_{v}\,\mathbf{M}_u - \partial^{\mathbf{s}\,\mathbf{v}}{}_{u}\,\mathbf{M}_v)$$

indicates that this 'force' is a convective momentum flow consisting of the convective momentum flow in the narrow sense and of the convective momentum flow due to deformation.

As a consequence of the antimetry of the 'permeability' this momentum flow is powerless

$$\mathbf{v}_u \mathbf{M}^C{}_u = 0 \, ,$$

as Lurie points out *expressis verbis* in a section on 'The structure of Lagrange's equations' (2002/309 f). In terms of geometrical and physical concepts it is orthogonal to the speed, it is a reactive internal flow of momentum 'between the degrees of freedom'

In Hamel's 'Theoretische Mechanik' the entry 'gyroscopic' does not occur in the subject index. And the author did not even find it in the context of rigid body dynamics, which is treated as usual only after global or aggregate mechanics of holonomic systems with holonomic generalised speed (1949/375-464). Schade, altogether 'the wrong way round', the other way is much more efficient.

19.2.3.4 GEOMETRICAL ALGEBRA

In geometrical and exterior or Grassmann algebra the antimetrical term

$$(\partial^{sv}{}_v \mathbf{M}_u - \partial^{sv}{}_u \mathbf{M}_v) = 2\,(\partial^{sv}{}_w \mathbf{M}_u)^{ant}$$

is called exterior or 'wedge' product,

$$(\partial^{sv}{}_v \mathbf{M}_u)^{ant} \equiv \partial^{sv}{}_v \wedge \mathbf{M}_u \, ,$$

the former name indicating the close relationship with the exterior vector or 'cross' product in three-dimensional space, the latter name being due to the symbolic notation used to avoid confusion with the 'cross' product.

As will be discussed for the particular product under discussion the name 'generalised rotor product' is adequate in direct analogy to the usage in continuum kinematics.

Under the conditions stated the exterior product is indeed the exterior vector product generalised to dimensions higher than three (Wikipedia):

> "The cross product and triple product in three dimensions each admit both geometric and algebraic interpretations. The cross product u × v can be interpreted as a vector which is perpendicular to both u and v and whose magnitude is equal to the area of the parallelogram determined by the two vectors. It can also be interpreted as the vector consisting of the minors of the matrix with columns u and v. The triple product of u, v, and w is geometrically a (signed) volume. Algebraically, it is the determinant of the matrix with columns u, v, and w. The exterior product in three-dimensions al-

> lows for similar interpretations. *In fact, in the presence of a positively oriented orthonormal basis, the exterior product generalizes these notions to higher dimensions.*" *Italics: MS.*

The advantages of geometrical algebra and the range of its applications are highlighted by the following quotation (Wikipedia):

> "In mathematical physics, a geometric algebra is a multilinear algebra described technically as Clifford algebra over a real vector space equipped with a non-degenerate quadratic form. Informally, *a geometric algebra is a Clifford algebra that includes a geometric product. This allows the theory and properties of the algebra to be built up in an intuitive, geometric way.* The term is also used in a more general sense to describe the study and application of these algebras: so Geometric algebra is the study of geometric algebras.
>
> *Geometric algebra is useful in physics problems that involve rotations*, phases or imaginary numbers. *Proponents of geometric algebra argue it provides a more compact and intuitive description of classical and quantum mechanics*, electromagnetic theory and relativity. Current applications of geometric algebra include computer vision, biomechanics and *robotics, and spaceflight dynamics*." *Italics: MS.*

19.2.3.5 Christoffel symbols

The expression for the gyroscopic 'force' is explicitly

$$\mathbf{M}^{C}{}_{u} = -\,\mathbf{v}_{v}\,(\partial^{s\,\mathbf{v}}{}_{v}\,\mathbf{I}_{u\,w} - \partial^{s\,\mathbf{v}}{}_{u}\,\mathbf{I}_{v\,w})\,\mathbf{v}_{w}$$

and the generalised tensor of the aggregate 'permeability'

$$\mathbf{Q}_{u\,v\,w} \equiv (\partial^{s\,\mathbf{v}}{}_{v}\,\mathbf{I}_{u\,w} - \partial^{s\,\mathbf{v}}{}_{u}\,\mathbf{I}_{v\,w})$$

called 'gyroscopic coefficients' (Lurie, 2002/ 310).

As the aggregate inertia, the aggregate permeability is a function of the mass distribution depending solely on the generalised location. The term 'permeability' has been introduced by the author in dealing with motions of bodies in fluids, using the sign convention stated (1962/12; 1964/16) in accordance with the terminology used in flow laws in fluid- and electrodynamics (Pohl, 1949/103 f).

Another format in terms of Christoffel 'symbols' of the first kind may be derived from the basic expression for the reactive momentum flow

$$\mathbf{M}^{C}{}_{u} = -\,\mathbf{v}_{v}\,\partial^{s\,\mathbf{v}}{}_{v}\,(\mathbf{I}_{u\,w}\,\mathbf{v}_{w}) + \partial^{s\,\mathbf{v}}{}_{u}\,(\mathbf{I}_{v\,w}\,\mathbf{v}_{v}\,\mathbf{v}_{w}/2)$$

$$= -\,(\partial^{s\,\mathbf{v}}{}_{v}\,\mathbf{I}_{u\,w} + \partial^{s\,\mathbf{v}}{}_{w}\,\mathbf{I}_{u\,v} - \partial^{s\,\mathbf{v}}{}_{u}\,\mathbf{I}_{v\,w})\,/2\,\mathbf{v}_{v}\,\mathbf{v}_{w}\,.$$

The term

$$[u, v; u] = (\partial^{s\,\mathbf{v}}{}_{v}\,\mathbf{I}_{u\,w} + \partial^{s\,\mathbf{v}}{}_{v}\,\mathbf{I}_{w\,u} - \partial^{s\,\mathbf{v}}{}_{u}\,\mathbf{I}_{v\,w})\,/2$$

is called Christoffel 'symbols' of the first kind for the generalised matrix of the aggregate inertia, 'for the matrix of the coefficients of the quadratic energy form' (Lurie, 2002/ 309 ff, /320 ff). Both names are felt to be not particularly enlightening, even confusing some fundamental differences between mathematics and physics, not meeting Goethe's and Maxwell's requirements.

In general Christoffel 'symbols' are not simple to use (Wikipedia):

> "In mathematics and physics, the Christoffel symbols, named for Elwin Bruno Christoffel (1829–1900), are coordinate-space expressions for the Levi-Civita connection derived from the metric tensor. In broader sense, the connection coefficients of an arbitrary (not necessarily metric) affine connection in a coordinate basis are often called Christoffel symbols. The Christoffel symbols may be used for performing practical calculations in differential geometry. *Unfortunately, the calculations are usually quite lengthy and complex, and require careful attention to detail. By contrast, the index-less, formal notation for the Levi-Civita connection is terse, and allows theorems to be stated in an elegant way, but requires more advanced techniques for practical calculations.* In many practical problems, the majority of Christoffel symbols are equal to zero." *Italics: MS.*

In the orthogonal Cartesian frames adopted throughout all the problems mentioned are *not* encountered and the index notation adopted 'allows theorems to be stated in an elegant way' and does *not* 'require advanced techniques for practical calculations'.

19.2.3.6 Convective transport

Re-writing the original law for the convective momentum flow

$$\mathbf{M}^{C}{}_{u} = -\,\mathbf{v}_{v}\,(\boldsymbol{\partial}^{sv}{}_{v}\,\mathbf{U}_{uw} - \boldsymbol{\partial}^{sv}{}_{u}\,\mathbf{U}_{vw})\,\mathbf{M}_{w}$$

results in the format

$$\mathbf{M}^{C}{}_{u} = -\,\mathbf{v}_{v}\,\mathbf{e}_{uvw}\,\mathbf{M}_{w}$$

with the generalised rotor or curl operator

$$\mathbf{e}_{uvw} \equiv (\mathbf{U}_{vr}\,\mathbf{U}_{uw} - \mathbf{U}_{ur}\,\mathbf{U}_{vw})\,\boldsymbol{\partial}^{sv}{}_{r}\,.$$

This law for the internal flow of momentum has been suggested by the author in direct analogy to the Euler equation for gyroscopes and the Euler-Lagrange equation for rigid bodies in three dimensional observation spaces.

The generalised rotor operator can be constructed for any generalised position and speed and for any number of degrees of freedom. The need to answer this question is felt after cursory inspection of the literature abounding with statements to the effect that the exterior vector product cannot be generalised *ad libitum*.

As the above equation shows explicitly the operator in question is a 'kinematical' operator in the context of the balance of aggregate momentum while the ε-'tensor', the Levi-Civita 'symbol' is a 'purely' mathematical concept by definition, but in mechanics is used as a kinematical concept, as generalised rotor operator, another source of confusion.

Thus after much deliberation the sloppy name 'generalised ε-operator' used in all the work of the author on rigid body motions in fluids has been discarded and is replaced here by the more adequate name 'generalised rotor operator', but the symbol **e** has been kept, indicating the relationship with an exterior product.

In the three-dimensional space the relation (Prager, 1961/27)

$$\varepsilon_{q\,i\,j}\,\varepsilon_{q\,k\,p} \equiv (\mathbf{U}_{j\,p}\,\mathbf{U}_{i\,k} - \mathbf{U}_{i\,p}\,\mathbf{U}_{j\,k})\ ,$$

holds for the ε-operator defined by

$$\varepsilon_{i\,j\,k} = +\,1 \text{ in case (u, v, w) even permutation,}$$

$$= -\,1 \text{ in case (u, v, w) odd permutation,}$$

$$= 0 \text{ in all other 21 cases,}$$

explicitly

$$\varepsilon_{1\,2\,3} = \varepsilon_{2\,3\,1} = \varepsilon_{3\,1\,2} = +\,1\ ,$$

$$\varepsilon_{1\,3\,2} = \varepsilon_{2\,1\,3} = \varepsilon_{3\,2\,1} = -\,1\ .$$

19.2.3.7 GENERALISED ROTOR OPERATOR

With the generalised rotor operator the balance of aggregate momentum, the explicit generalised Lagrangean equation assumes the intuitive form

$$\mathrm{d}_t\,\mathbf{M}_u + \mathbf{e}_{u\,v\,w}\,\mathbf{v}_v\,\mathbf{M}_w = \mathbf{M}^M{}_u + \mathbf{M}^T{}_u + \mathbf{M}^P{}_u$$

referred to an orthogonal reference frame fixed in the space of generalised speed. *As all magnitudes involved are referred to the same frame explicit reference to the frame in the notation is conveniently suppressed.*

This balance is for short called the Euler-Lagrangean, also generalised Euler equation or Kane[TR]'s generalised dynamical equation (Kane[TR], 1985/159). Usually the equation is derived for systems of mass points and of rigid bodies only.

As long as the generalised rotor operator cannot be constructed the Euler-Lagrangean equation is still not really useful. The following conditions have to be observed in the construction in any particular case.

The most fundamental condition is derived from the required invariance of the aggregate momentum balance under changes of frames for a given choice of generalised position and speed. The explicit equation

$$\mathbf{M}^{C\,\mathrm{transf}}{}_{p} = \mathbf{T}_{p\,u}\,\mathbf{M}^{C}{}_{u}$$

$$= -\,\mathbf{T}_{p\,u}\,\mathbf{e}_{u\,v\,w}\,\mathbf{T}_{v\,q}\,\mathbf{T}^{-1}{}_{q\,s}\,\mathbf{v}_{s}\,\mathbf{T}^{-1}{}_{w\,r}\,\mathbf{T}_{r\,t}\,\mathbf{M}_{t}$$

$$= -\,\mathbf{e}_{p\,q\,r}\,\mathbf{v}^{\mathrm{transf}}{}_{q}\,\mathbf{M}^{\mathrm{transf}}{}_{r}$$

with the transformed speed

$$\mathbf{v}^{\mathrm{transf}}{}_{p} = \mathbf{T}^{-1}{}_{q\,s}\,\mathbf{v}_{s}$$

and the transformed momentum

$$\mathbf{M}^{\mathrm{transf}}{}_{r} = \mathbf{T}_{r\,t}\,\mathbf{M}_{t}$$

implies

$$\mathbf{e}_{p\,q\,r} = \mathbf{T}_{p\,u}\,\mathbf{e}_{u\,v\,w}\,\mathbf{T}_{v\,q}\,\mathbf{T}^{-1}{}_{w\,r}\,.$$

The generalised rotor operator is invariant and consequently unit-free. Accordingly the generalised rotor operator is not a generalised tensor, as the ordinary $\boldsymbol{\varepsilon}$-operator in three-dimensional space is not a tensor contrary to the *false* terminology found in all textbooks and on all websites inspected, sometimes with reference to the problem (Prager, 1961/26-29).

This implies that the derivatives with respect to the position must be unit-free and this is possible only if the pertinent components of the generalised positions are unit-free, typically angular displacements. Further the pertinent components of the speed must be rotational speeds.

Two instances come to mind immediately: rotational motions of rigid bodies at rest and in translational motion *in vacuo* and in fluids in three dimensional observation spaces. In the first case the operator is the ε-operator proper, providing the ordinary exterior vector product. In the second case the generalised rotor operator has ε-operators proper as components only where the rotational speed is involved.

Further conditions are obtained observing the fact that the gyroscopic 'force' is normal to the generalised speed and to the generalised momentum

$$\mathbf{v}_{u}\,\mathbf{M}^{C}{}_{u} = \mathbf{v}_{u}\,\mathbf{e}_{u\,v\,w}\,\mathbf{v}_{v}\,\mathbf{M}_{w} = 0\,,$$

$$\mathbf{M}_{u}\,\mathbf{M}^{C}{}_{u} = \mathbf{M}_{u}\,\mathbf{e}_{u\,v\,w}\,\mathbf{v}_{v}\,\mathbf{M}_{w} = 0$$

The first condition is that of vanishing power already mentioned: in formal changes of frames no power is involved.

A third condition is that of vanishing reactive momentum flow in case of spherical aggregate inertia tensor

$$\mathbf{0}_{u} = \mathbf{e}_{u\,v\,w}\,\mathbf{v}_{v}\,\mathbf{v}_{w}\,.$$

There are no gyroscopic 'forces' in these cases. The most prominent instance is that of translational motions of rigid bodies.

The three conditions stated imply the following three conditions of antimetry

$$\mathbf{e}_{u\,v\,w} = -\,\mathbf{e}_{v\,u\,w}\,,$$

$$\mathbf{e}_{u\,v\,w} = -\,\mathbf{e}_{w\,v\,u}\,,$$

$$\mathbf{e}_{u\,v\,w} = -\,\mathbf{e}_{u\,w\,v}\,.$$

As a result of the conditions stated most of the components of the generalised rotor operator are zero.

19.2.3.8 PRINCIPAL GENERALISED SPEED

For theoretical and practical purposes it may be useful to construct a frame of generalised speed in which the matrix of aggregate inertia reduces to a diagonal matrix, the frame being called principal frame of speed.

The condition to be observed is that the kinetic energy does not change

$$\mathbf{v}^{\text{prin}}{}_{s}\,\mathbf{I}^{\text{prin}}{}_{s\,o}\,\mathbf{v}^{\text{prin}}{}_{o} = \mathbf{v}_{r}\,\mathrm{T}^{-1}{}_{r\,s}\,\mathrm{T}_{s\,u}\,\mathbf{I}_{u\,v}\,\mathrm{T}_{v\,o}\,\mathrm{T}^{-1}{}_{o\,p}\,\mathbf{v}_{p}\,.$$

Accordingly the principal speed is

$$\mathbf{v}^{\text{prin}}{}_{o} = \mathrm{T}^{-1}{}_{o\,p}\,\mathbf{v}_{p}$$

if the principal inertia is

$$\mathbf{I}^{\text{prin}}{}_{s\,o} = \mathrm{T}_{s\,u}\,\mathbf{I}_{u\,v}\,\mathrm{T}_{v\,o}\,.$$

It is noticed that the transformation necessary can only be performed after the matrix of inertia has been identified in a preliminary speed frame. Even if the inertia does not change in speed space or even if it is invariant in speed space the transformation to a principal frame often is not worth the labour.

For theoretical and qualitative discussions the transformation is particularly useful if it permits to separate the balance of momentum into simpler ones, as in case of case of rigid body motions *in vacuo* into balances for translational and rotational momentum.

The situation is exactly the same with any transformation into 'principal' reference frames, simplifications of models can be achieved only *a posteriori*. Examples are the search for the 'relevant' parameters in dimensional analysis and the Karhunen-Loève theorem permitting 'principal' component analysis.

19.2.4 Partial energy balances: appraised

"Erst wenn Du weißt, was Du tust,
kannst Du tun, was Du willst."

Moshe Feldenkrais zur Begründung seiner 'Bewegungslehre'.

19.2.4.1 Complete state space model

Before laws for the aggregate 'forces', the aggregate momentum diffusion, reversible storage, irreversible dissipation and production are being considered it is appropriate to review the results obtained so far.

The extremely concise state space model for simple forcibly driven holonomic systems of bodies of classical matter with non- holonomic generalised speed consists of:

- the dynamical state equation, the balance of aggregate momentum, the Euler-Lagrangean equation

$$\mathrm{d}_t \mathbf{M}_u + \mathbf{e}_{uvw} \mathbf{v}_v \mathbf{M}_w = \mathbf{M}^M{}_u + \mathbf{M}^T{}_u + \mathbf{M}^P{}_u ,$$

- for the momentum

$$\mathbf{M}_u = \mathbf{I}_{uv} (\mathbf{s}) \mathbf{v}_v$$

- and the kinematical state equation

$$\mathrm{d}_t \mathbf{s}_u = \mathbf{K}_{uv} (\mathbf{s}) \mathbf{v}_v$$

constituting the position space, explicitly stating the kinematics of the system of bodies.

At this stage it needs to be realised that the theory of state space models, abstract meta-mechanics is prerequisite for an adequate appraisal of this model. Only both state equations together provide a complete state space model. The situation is exactly the same as in non-relativistic quantum mechanics, where Bohm's kinematic state equation complements Schrödinger's wave equation, the dynamic state equation.

On the one hand, in view of the world around us, our planetary system and our galactic system, the state space model developed appears to be the most natural, describing 'the universe' as a generalised gyroscope on whatever level of aggregation required for the purpose at hand.

On the other hand, the success of Newton's 'System of the World' (PM, Book 3) is ample evidence that for the purposes of celestial mechanics the adoption of holonomic generalised speed is sufficient. Similarly in technical mechanics limitation to holonomic generalised speed is often sufficient, the motions of gyroscops usually being considered and treated as subject of 'higher mechanics' but surprisingly without reference to analytical mechanics (Szabó, 1956/88-98).

19.2.4.2 PARTIAL ENERGY BALANCES: *DERIVED*

In view of the 'incredible' derivations of the Euler Lagrangean equation, still following Lagrange derivation first published 1788, and statements concerning the kinematical state equation even in advanced textbooks it is of foremost importance to note the fact that the state space model has been arrived at without referring to 'variations' or any other 'principles' solely by direct weighted integration of Cauchy's universal local balance of momentum for simple forcibly driven systems of bodies of ponderable matter.

The integration using partial velocities as weights avoids all unnecessary confused and confusing considerations and elucidates the essentially simple implications, which need to be clearly understood by engineers for sensible and responsible applications to real problems. The foregoing derivation takes all the romance out of the theory and is so simple that even high school students, Gymnasiasten, can follow and even remember it for their future lifes.

Derivations in advanced textbooks, usually starting from Hamilton's principle, may look more advanced but have been found inadequate, not only for the purposes of this treatise, and obscuring all the essentially simple basic issues (Lanczos, 1986/111-160; Kane^{TR}, 1985/50-56; Gummert, 1994/740-747; Scheck, 2003/79-169). The detailed sections in Morgenstern's lectures on theoretical mechanics and in Kane^{TR}'s and Lurie's textbooks look close to present exposition, but are felt to lack the intuitive appeal and reference to physics.

The problems faced in reading 'professional' derivations are illustrated by Lurie's sentence introducing the derivation of the Lagrangean equation, the latter to be considered here as a special instance in due course (2002/303):

> "Differential equations of motion for the generalised coordinates can be obtained easily with the help of Lagrange's central equation. The equations will be derived twice here. The first derivation will assume that the operations d and δ are not interchangeable, while the second one will assume that the operations are interchangeable. In the first case, i.e. if $d\,\delta \neq \delta d$, it is necessary to use Lagrange's central equation."

The purpose of the present exercise to get along without variations has been the same as the purpose of Morgenstern and Szabó (1961/71 ff):

> " … um der verbreiteten Betrachtung an 'kleinen Elementen', die oft so kunstvoll sind, daß sie nur der Kenner verstehen kann, weiterhin aus dem Wege zu gehen."

Most 'simple minded' textbooks do not treat the case of non-holonomic generalised speed. Consequently rigid body motion has to be treated without reference to the foregoing chapter on analytical mechanics. Surprising is that in the textbook of Reckling and Gummert the present approach is not followed although in the beginning Cauchy's universal local balance of momentum is considered as the sole fundament of mechanics.

Concerning the Euler-Lagrangean form of the momentum balance Lurie's claim is nearly the same advantages as that of Kane [TR], even in wording (2002/391):

> "These equations differing from Lagrange's equations by introducing quasi-velocities instead of the generalised velocities were derived by Boltzmann … and Hamel … at approximately the same time. It was Hamel who suggested the above name für these equations. The equations of motion used by Voronets also deal with the quasi-velocities, however their form differs slightly from the Euler-Lagrange equations.
>
> The Euler-Lagrange equations are mainly used for non-holonomic systems and they were suggested to this aim as is seen from the titles … However their significance is not limited to these special problems since these equations can considerably simplify the form and the process of constructing equations of motion for holonomic systems, too. We will have an opportunity to convince ourselves as to how fruitful it is to apply the Euler-Lagrange equations to some problems of dynamics of multi-body systems."

For exactly these reasons the author has used this equation with great advantage during his whole professional life.

Incidentally, Hamel did not use the name 'Euler-Lagrangean' equation but mentions (1949/481):

> "Wir haben diese Gleichungen in Bd. 50 der Zeitschrift für Mathematik und Physik 1904 als 'Lagrange-Eulersche' bezeichnet, weil sie … "

Hamel further mentions that the equation has also been derived by Volterra and Woronetz and that Poincaré and Boltzmann came close to it.

The abstract of Hamel's paper 'Über nichtholonome Systeme' is of interest here (1924):

> "Den Anstoß zu dieser Arbeit gaben zwei Noten des Herrn Ivan Tzénoff: 'Sur les équations du mouvement des systèmes matériels non holonomes', die 1920 in Liouvilles Journal und soeben 1924 in diesen Annalen (Math.

Annalen 91) erschienen sind. Herr Tzénoff leitet Bewegungsgleichungen ab, die einen Mischtyp aus Lagrange und Appel darstellen; die Rechnung kann erheblich vereinfacht werden, so daß man das Resultat sofort einsieht:das soll in § 1 geschehen.

Weiter vergleicht Herr Tzénoff seine Gleichungen mit denen, die Herr Woronetz in seiner Arbeit (Math. Annalen 70): 'Über die Bewegung eines starren Körpers, der ohne Gleitung auf einer beliebigen Fläche rollt' 1911 in § 5 abgeleitet hat und die inhaltlich mit den von mir in meiner Habilitationsschrift 1903 (auch in der Zeitschrift für Math. und Physik 1904): 'Die Lagrange-Eulerschen Gleichungen der Mechanik' sowie in der Annalenarbeit (Math. Annalen 59) 'Über die virtuellen Verschiebungen in der Mechanik' gegebenen Bewegungsgleichungen übereinstimmen, nur daß meine Gleichungen erheblich weiterreichen, indem sie die Verwendung beliebiger nichtholonomer Geschwindigkeitsparameter gestatten. In der Form aber sind meine Gleichungen so übersichtlich wie die Lagrangeschen, was man von denen des Herrn Woronetz nicht behaupten kann. Da meine Arbeiten offenbar wenig bekannt sind – Herr Tzénoff nennt sie nicht –, möchte ich in § 2 kurz zeigen, wie die Gleichungen des Herrn Woronetz als Spezialfall in meinen enthalten sind."

According to the present derivation 'auch Hamel's Rechnung kann erheblich vereinfacht werden, so daß man das Resultat sofort einsieht' as well as the underlying assumptions.

19.2.4.3 BASIC WORK OF THE AUTHOR

In writing his doctoral thesis on 'a general equation for the motions of rigid bodies in fluids' the author has constructed and used this state space model 'of course' without knowing that the case considered was only the simplest instance of the much more general theory (Schmiechen, 1962; 1964).

And, even worse, 'of course' he was not told so, but he was urged to delete the essential introductory sentence, which 'of course' he did not:

"Zur quantitativen Beschreibung der Bewegungen eines starren, nicht notwendig homogenen Körpers in einem dreidimensionalen Euklidischen Beobachtungsraum werden als Bewegungsmengen oder – quantitäten, d. h. als extensive Bewegungsgrößen der Impuls und das Impulsmoment des Körpers … eingeführt."

In another case the equation was felt to be just crumbs of chalk at the black board. The reason was and still is that most hydrodynamicists are not concerned with conceptual problems but with the interpretation of concepts analytically or by physical or numerical experiments. The concepts are 'given' to them.

But concepts are not presents of heaven, of inspiration or enlightenment. That they are meaningful and can be interpreted only in the context of adequate models generated *ad hoc* for the purpose at hand is an idea not usually entertained by 'experts', while for Goethe this was evident as has been mentioned (GA 17/723):

> "Das Höchste wäre: zu begreifen, daß alles Faktische schon Theorie ist."

The explicit theory of state space models has been developed only later during leaves of absence from the duties at VWS for research at the MIT (1968/69) and at the University of Tokyo (1973).

19.2.4.4 'EPISTEMOLOGICAL RE-EXAMINATION'

At the other extreme end of research a current study by Wahsner may be of interest (MPIWG, 2002/39 f):

> "The role played by the conflict between mechanism and organism in the epistemological foundation and reflection of modern natural science is investigated. Continuing earlier systematic work, the approach, or rather the 'cunning reason', is analyzed which physics has found to capture motion by a principle allowing its calculation and measurement, although motion has, according to Hegel, to be philosophically conceptualized as a contradiction in real existence. This principle comprises essential features of Kant's concept of mechanism but also of Hegel's and Kant's concept of organism. *It is planned to include an epistemological re-examination of analytical mechanics from this perspective.*" *Italics: MS.*

19.2.5 Special case: force free motions

19.2.5.1 EQUILIBRIUM CONDITION

The case of force free motions

$$d_t (\partial^v{}_u E) - K_{vu} \partial^s{}_v E = \mathbf{0}_u ,$$

explicitly

$$\partial^v{}_t \mathbf{M}_u + \mathbf{e}_{uvw} \mathbf{v}_v \mathbf{M}_w = \mathbf{0}_u$$

in terms of aggregate momentum

$$\mathbf{M}_u = \mathbf{I}_{uv} (\mathbf{s}) \mathbf{v}_v$$

is of particular interest.

In this case the aggregate 'forces'

$$\mathbf{M}^M{}_u + \mathbf{M}^T{}_u + \mathbf{M}^P{}_u = \mathbf{0}_u ,$$

the aggregate diffusive flow and the aggregate productions of aggregate momentum in the system balance each other. Often the special case of separately vanishing aggregate 'forces', the case of

$$\mathbf{M}^{M}{}_{u} = \mathbf{0}_{u} : \quad \text{freely moving systems}$$

$$\mathbf{M}^{T}{}_{u} = \mathbf{0}_{u} : \quad \text{of rigid bodies}$$

$$\mathbf{M}^{P}{}_{u} = \mathbf{0}_{u} : \quad \text{in inertial spaces}$$

is being considered.

19.2.5.2 Minimum principle

According to the derivation of the balance of aggregate momentum as partial energy balances

$$\partial^{v}{}_{u}\,(d_{t}\,E) = d_{t}\,(\partial^{v}{}_{u}\,E) - \partial^{s\,v}{}_{u}\,E = \mathbf{0}_{u}$$

in case of vanishing *net* force the time rate of change of the kinetic energy has an extremum

$$d_{t}\,E = \text{extr}\,.$$

'By definition' the implicit and explicit balance of aggregate momentum is the conditions for that extremum.

As there are no 'forces' 'doing any work' the energy remains unchanged. Accordingly the value of the extremum is zero

$$d_{t}\,E = 0\,.$$

and the kinetic energy of the system does not change in the case considered

$$E = \text{const}\,.$$

19.2.5.3 Variational principle

In any textbook on analytical mechanics or including a chapter on analytical mechanics the generalised Lagrangean equation, the implicit Euler-Lagrangean equation

$$d_{t}\,(\partial^{v}{}_{u}\,E) - \partial^{s\,v}{}_{u}\,E = \mathbf{0}_{u}$$

is shown to be the Eulerian equation corresponding to the variational problem

$$\delta \int_{\Delta t} E\,dt = \int_{\Delta t} \delta\,E\,dt = \text{extr}\,,$$

integration being taken over any time interval, leaving the values of the energy unchanged at the beginning and at the end of the interval.

If the time interval is conveniently taken to be infinitesimal

$$\Delta t = dt$$

the integral reduces to

$$\delta\, E \int_{\Delta t} dt = \delta\, E = 0\,,$$

at any instant in time. Consequently, the total variation of the integral, the extremum vanishes as well

$$\delta \int_{\Delta t} E\, dt = 0\,.$$

In many textbooks the Lagrangean equation, is derived from the variational problem, stated as Hamilton's principle (Szabó, 1956/84-88; Gummert, 1994/740-744). Usually this is done only for the case of holonomic speed but including aggregate momentum productions due to potentials, the elastic potential and/or the gravitational potential.

The present exposition shows that these formal approaches obscure all the fundamental implications the explicit knowledge of which is essential for understanding and applying the theory sensibly and efficiently. The fact that Lagrange's 'Méchanique analytique' does not contain figures does not imply that one cannot have a clear 'Anschauung' of 'what is going on' in speed and position space or, to paraphrase Goethe's dictum, (Eckermann, 1911/140: 20.12.1826):

> "Um die Phänomene der *Mechanik* zu begreifen, gehört weiter nichts als ein reines Anschauen und ein gesunder Kopf; allein beides ist freilich seltener als man glauben sollte." *Italics:* Farbenlehre *in the original.*

19.2.6 Special case: potential 'forces'

In case the 'forces' can be derived from an elastic potential

$$\mathbf{M}^{S}{}_{u} = -\,K_{v\,u}\,\partial^{s}{}_{v}\,E^{S} = -\,\partial^{s\,v}{}_{u}\,E^{S}$$

and/or a gravitational potential

$$\mathbf{M}^{P}{}_{u} = -\,K_{v\,u}\,\partial^{s}{}_{v}\,E^{P} = -\,\partial^{s\,v}{}_{u}\,E^{P}\,,$$

respectively, the foregoing reasoning remains the same if instead of the kinetic energy the 'kinetic' or Lagrangean potential

$$E = E - E^{M} - E^{P}$$

is being introduced.

Under the condition

$$\partial^{v}{}_{u}\,E^{M} = \mathbf{0}_{u}$$

$$\partial^{v}{}_{u}\,E^{P} = \mathbf{0}_{u}$$

the generalised Lagrange equation becomes

$$d_t\,(\partial^v{}_u\,E^L) - \mathbf{K}_{v\,u}\,\partial^s{}_v\,E^L = \mathbf{0}_u\,,$$

and this is nothing else but the condition

$$\partial^v{}_u\,d_t\,E^L = \mathbf{0}_u$$

for an extremum of the derivative of the kinetic potential, the value

$$d_t\,E^L = 0$$

and finally

$$E^L = \text{const}\,.$$

Transformations of the mechanical potential will be discussed in the context of the special case of holonomic generalised speed, in case of degenerate kinematics.

As has been noted before the Lagrangean equation is equivalent to Hamilton[WR]'s principle

$$\int_T E^L\,dt = \text{extr}$$

or, accordingly,

$$\delta \int_T E^L\,dt = 0$$

for the kinetic potential is introduced before. This principle has been subject of many speculations.

For stable motions the extremum is a minimum, the second variation being positive

$$\delta^2 \int_T E^L\,dt > 0\,.$$

19.2.7 'Energetik'

At a meeting of the *Gesellschaft deutscher Naturforscher und Ärzte* at Lübeck in 1895 a battle was fought between the *Bildtheorie*, represented by Boltzmann, and the *Energetik*, represented by Oswald. Scheibe[E] quotes a supposedly neutral report by Sommerfeld, who among other famous physicists was present at the meeting (2006/105 f):

"Das Referat für die Energetik hatte Helm - Dresden; hinter ihm stand Wilhelm Ostwald, hinter beiden die Naturphilosophie des nicht anwesenden Ernst Mach. Der Opponent war Boltzmann, sekundiert von Felix Klein. Der Kampf zwischen Boltzmann und Ostwald glich, äußerlich und innerlich, dem Kampf des Stiers mit dem geschmeidigen Fechter. Aber der Stier besiegte diesmal den Torero trotz all seiner Fechtkunst. Die Argumente

> Boltzmanns schlugen durch. Wir damals jungen Mathematiker standen alle auf der Seite Boltzmanns; es war uns ohne weiteres einleuchtend, daß aus der einen Energiegleichung unmöglich die Bewegungsgleichungen auch nur eines Massenpunktes, geschweige denn eines Systems von beliebigen Freiheitsgraden gefolgert werden könnten."

Subsequently ScheibeE mentions (2006/106):

> "Ostwalds eigener Vortrag stand unter dem Titel 'Die Überwindung des wissenschaftlichen Materialismus'. Darunter versteht er die Lehre, 'daß die Dinge sich aus bewegten *Atomen* zusammensetzen und daß diese Atome und die zwischen ihnen wirkenden *Kräfte* die letzten Realitäten seien, aus denen die einzelnen Erscheinungen bestehen'. Ostwalds Anliegen bestand darin zu zeigen, '*daß diese so allgemein angenommene Auffassung unhaltbar ist*'.

At the level of aggregate mechanics even the continuum model is 'left behind', being itself an aggregate description of the molecular structure of matter, to forget about the nuclear structure necessary to explain inertia and gravity.

Hertz in the 'Introduction' to his 'Prinzipien der Mechanik' is discussing the approach to mechanics based on energy at length and in summarising discards it (1956/40):

> "In conclusion, let us glance once more at the three images of mechanics which we have brought forward, and let us try to make a final and conclusive comparison between them. After what we have already said, we may leave the second image [based on energy] out of consideration."

The interesting point in the present context is that Sommerfeld's firm statement 'daß aus der einen Energiegleichung unmöglich die Bewegungsgleichungen auch nur eines Massenpunktes, geschweige denn eines Systems von beliebigen Freiheitsgraden gefolgert werden könnten' is too strong.

Neither Sommerfeld nor Hertz noticed that the Euler-Lagrangean equation is 'nothing but' the partial energy balances for any generalised speed convenient for the purposes at hand. The derivation given shows the implications and limitations.

As has been mentioned the traditional discussion suffers from the confusion of the concepts of 'balance' and 'conservation' of energy. Theories not accounting for energy dissipation, though figuring prominently in theoretical physics, are only of limited interest in classical mechanics as practised by engineers. It is felt that most exposition are lacking intuitive appeal and adequate reference to physics, 'hiding' the essentials, typically limiting assumptions of ideal constraints etc behind details of minor importance.

EVALUATIONS/ASSESSMENTS

CONCLUSIONS

The partial energy balances, the Euler-Lagrangean equation in particular, will be applied in the following chapters in dealing with special classes of bodies of matter, rigid bodies in vacuo and in ideal fluids.

Before that is being done laws for the aggregate forces will be considered and an attempt will be made to reconstruct the famous 'variational principles of mechanics', which have attracted so much interest, absorbed so much brain power and aroused so many speculations in the history of mechanics and other sciences.

19.3 Aggregate 'forces'

PROBLEMS

So far the changes of aggregate momentum, of the partial powers have been dealt with.

MODELS

These changes are due to the aggregate forces conceived as the partial powers.

GOALS/PLANS

The goal is to elaborate this model as far as necessary for the purposes of this treatise.

19.3.1 Aggregate momentum diffusion

19.3.1.1 GENERAL LAW

On first sight the aggregate momentum diffusion into a continuous body of matter

$$\mathbf{M}^{M}{}_{\mathbf{u}} \equiv \partial^{v}{}_{u} E^{M} \equiv \partial^{v}{}_{u} \int_{A} v_{i}\, \sigma_{ij}\, dA_{j} = \int_{A} \mathbf{v}_{iu}\, \sigma_{ij}\, dA_{j}$$

looks rather prohibitive. But on the aggregate level it can be interpreted most easily if conceived as diffusive momentum flow from the environment of the body of matter, and if this environment is conveniently conceived as constituted by other bodies of matter subject to formally the same space state model with their own generalised speed and position spaces, in general different in dimensions from the dimension of the body under investigation.

Accordingly a system of state space models has to be considered

$$d_t \mathbf{M}^a{}_u + \mathbf{e}_{u v w} \mathbf{v}^a{}_v \mathbf{M}^a{}_w = \mathbf{M}^{M a}{}_u + \mathbf{M}^{T a}{}_u + \mathbf{M}^{P a}{}_u ,$$

for the momentum

$$\mathbf{M}^a{}_u = \mathbf{I}^a{}_{u v} (\mathbf{s}) \mathbf{v}^a{}_v$$

and the kinematical state equation

$$d_t \mathbf{s}^a{}_u = \mathbf{K}^a{}_{u v} (\mathbf{s}) \mathbf{v}^a{}_v ,$$

for any individual body of matter

$$a, b, c = 0, 1, \ldots, n_b$$

with f_b degrees of freedom

$$u, v, w = 1, 2, \ldots, f_b .$$

The system of bodies of matter is now defined by its antimetrical matrix of diffusive flows of momentum between any two bodies, and thus total the diffusive flow into any body is obtained as the sum of all 'partial' diffusive flows

$$\mathbf{M}^{M a}{}_u = \Sigma_b \mathbf{M}^{M a b}{}_u .$$

The matrix is antimetrical due to the continuity of the diffusive flow across the boundary between any two bodies of matter.

This model permits to generate in a consistent fashion models for the motions of systems of bodies of matter of any intricacy depending on the structure of the matrix of diffusive flows.

Specifications, usually simplifying as appropriate for the purpose at hand, of this general model result in the particular branches and twigs of mechanics. Very prominent among those special branches are the theory of systems of rigid bodies and the theory of elasto-statics of systems at rest. Of general interest are only the simplest models.

19.3.1.2 SINGLE BODY WITH INFINITE ENVIRONMENT

The simplest model considered here is that of two bodies or rather three, a body of matter and its material environment, the latter having a boundary at infinity.

Accordingly the diffusive flows balance as follows

$$\mathbf{M}^{M\,body}{}_u = \mathbf{M}^{M\,body\,env}{}_u$$

$$\mathbf{M}^{M\,env}{}_u = \mathbf{M}^{M\,env\,body}{}_u + \mathbf{M}^{C\,env\,inf}{}_p + \mathbf{M}^{M\,env\,inf}{}_p$$

with the condition of continuity

$$\mathbf{M}^{\text{M body env}}{}_{u} = -\,\mathbf{M}^{\text{M env body}}{}_{u}\,.$$

and accounting for the fact that across the boundary at infinity convective momentum transport into the environment may take place.

Accordingly the complete law of the aggregate momentum diffusion into the body of interest becomes

$$\begin{aligned}\mathbf{M}^{\text{M body}}{}_{u} &= -\,\mathbf{M}^{\text{M env body}}{}_{p} + \mathbf{M}^{\text{M inf}}{}_{p} \\ &= \mathbf{M}^{\text{C inf}}{}_{p} + \mathbf{M}^{\text{M inf}}{}_{p} + \mathbf{M}^{\text{T env}}{}_{p} + \mathbf{M}^{\text{P env}}{}_{p} \\ &\qquad - \partial^{\mathbf{v}}{}_{t}\,\mathbf{M}^{\text{env}}{}_{p} - \boldsymbol{\varepsilon}_{p\,q\,r}\,\mathbf{v}^{\text{env}}{}_{q}\,\mathbf{M}^{\text{env}}{}_{r}\,.\end{aligned}$$

19.3.1.3 Forcibly driven environment

In the simplest case the environment has the same (number of) degrees of freedom as the body of matter under consideration and is forcibly driven by that body. In this simple case the state equations of the body and its environment can be added thus resulting in the state equation of the total system

$$\begin{aligned}&\mathrm{d}_t\,\mathbf{M}^{\text{body + env}}{}_{u} + \mathbf{e}_{u\,v\,w}\,\mathbf{v}_{v}\,\mathbf{M}^{\text{body + env}}{}_{w} \\ &\qquad = \mathbf{M}^{\text{C inf}}{}_{u} + \mathbf{M}^{\text{M inf}}{}_{u} + \mathbf{M}^{\text{T body + env}}{}_{u} + \mathbf{M}^{\text{P body + env}}{}_{u}\,.\end{aligned}$$

with the momentum

$$\mathbf{M}^{\text{body + env}}{}_{u} = \mathbf{I}^{\text{body + env}}{}_{u\,v}\,\mathbf{v}_{v}$$

and the kinematical state equation

$$\mathrm{d}_t\,\mathbf{s}_{u} = \mathbf{K}_{u\,v}\,\mathbf{v}_{v}\,.$$

This is exactly the same state space model as before the effects of the environment explicitly accounted by the 'added' inertia

$$\mathbf{I}^{\text{body + env}}{}_{u\,v} = \mathbf{I}^{\text{body}}{}_{u\,v} + \mathbf{I}^{\text{env}}{}_{u\,v}$$

and the added 'forces'

$$\mathbf{M}^{\text{T body + env}}{}_{u\,v} = \mathbf{M}^{\text{T body}}{}_{u\,v} + \mathbf{M}^{\text{T env}}{}_{u\,v}$$

$$\mathbf{M}^{\text{P body + env}}{}_{u\,v} = \mathbf{M}^{\text{P body}}{}_{u\,v} + \mathbf{M}^{\text{P env}}{}_{u\,v}$$

and last, but not least, the convective and diffusive momentum flow into the environment across its 'outer' boundary.

This is the model naval architects are referring to, for simplicity usually talking about rigid bodies in incompressible water

$$\mathrm{d}_t\,\mathbf{M}^{\text{body + env}}{}_{u} + \mathbf{e}_{u\,v\,w}\,\mathbf{v}_{v}\,\mathbf{M}^{\text{body + env}}{}_{w}$$

$$= \mathbf{M}^{C\,inf}{}_{u} + \mathbf{M}^{M\,inf}{}_{u} + \mathbf{M}^{T\,env}{}_{u} + \mathbf{M}^{P\,body\,+\,env}{}_{u}\ .$$

with the momentum

$$\mathbf{M}^{body\,+\,env}{}_{u} = (\mathbf{I}^{body}{}_{u\,v} + \mathbf{I}^{env}{}_{u\,v})\ \mathbf{v}_{v}\ .$$

While in rigid bodies the force due to deformation vanishes, the same is not the case in incompressible water, where dissipation 'takes place' and vortex streets and wave trains carry momentum across the boundary at infinity.

This model has been used to simulate a boomerang (Schmiechen, unpublished manuscript and BASIC programme, 1984) and motions of robots in flows (Schmiechen, 1992.u.3). While a boomerang is quite naturally envisaged as a gyroscope, robots in flows, maybe sonar antennas or swimmers, are not usually seen as generalised gyroscopes in multidimensional position spaces, although that is the most efficient model.

19.3.1.4 'EXTENDED' DYNAMICS

If the environment, as will usually be the case, has more degrees of freedom than the system of bodies of matter under investigation the state space models cannot be simply added, but the additional degrees of freedom have to be accounted for in a way adequate for the purpose at hand.

Essentially the additional states describe eigen-motions of the environment excited by the motions body. In fluids these eigen-motions are typically vortex streets and wave trains following their own laws.

Earlier this problem has been treated as straightforward instance of the general theory of meta-mechanics and applications of linear models to hydromechanical systems have been subject of extended theoretical and numerical studies as well as physical experiments on model scale (Schmiechen, 1974, 1992.u.3).

This line of work will not be reconstructed in detail in the present treatise results having been published elsewhere, some on the website of the author, but mostly buried in reports, in computer programs and in on-going correspondence on fundamentals and details of conceptual and computational solutions.

The basic problem is to define the state of the environment. If the future velocity field in the environment does not only depend on present generalised motions of the driving body, the 'past' history of its generalised motion has to be accounted for.

Describing the generalised speed by a Taylor series

$$\mathbf{v}_{u}\,(t) = \Sigma_{p}\,\mathbf{v}^{(p)}{}_{u}\,(t)$$

the generalised state of the environment can be described by the instantaneous derivatives of the generalised speed

$$\mathbf{s}_{u\,p}(t) = \mathbf{v}^{(p)}{}_{u}(t)\,.$$

19.3.2 Aggregate momentum productions

19.3.2.1 GENERAL ASPECTS

A special class of models of great importance in science and engineering are systems for which the aggregate momentum production

$$\mathbf{M}^{P}{}_{u} \equiv \int_{m} \boldsymbol{\nu}_{i\,u}\, f_{i}\, dm\,,$$

may be 'derived from' a potential

$$\mathbf{M}^{P\,s}{}_{u} = -\,\partial\, E^{P} / \partial\, \mathbf{s}_{u}\,.$$

constituted by the laws discussed earlier. This case is the one usually referred to in textbooks on theoretical physics in view of the classical applications in celestial and quantum mechanic.

In case of non-holonomic speed considered here this aggregate 'force' has to be transformed from position space into the speed space resulting in the law

$$\mathbf{M}^{P\,v}{}_{u} = \mathbf{M}^{P\,s}{}_{v}\, \mathbf{K}_{v\,u}\,.$$

19.3.2.2 UNIFORM PRODUCTION FIELD

In terrestrial mechanics the special case of a uniform field of momentum production in the observation space

$$f_{i} = g_{i} = \text{const}$$

is of great interest and importance. In this case the law for the gravity 'force' becomes simply

$$\mathbf{M}^{P\,s}{}_{u} = g_{i} \int_{m} \boldsymbol{\nu}_{i\,u}\, dm \equiv g_{i}\, \mathbf{m}_{i\,u}$$

with the aggregate mass or 'gravity' of the body of matter

$$\mathbf{m}_{i\,u} \equiv \int_{m} \boldsymbol{\nu}_{i\,u}\, dm$$

depending on its generalised position.

Nobody will ever seriously confuse the aggregate mass or 'gravity' with the aggregate inertia

$$\mathbf{I}_{u\,v} \equiv \int_m \boldsymbol{\nu}_{i\,u}\, \boldsymbol{\nu}_{i\,v}\, dm\ .$$

as is usual in elementary mechanics. The discussion of Atwood's machine has already provided an example of an aggregate system with inertia and gravity differing.

19.3.2.3 MOMENTUM DISSIPATION AND STORAGE

While the derivations of laws for the aggregate momentum diffusion and production are rather straightforward, the laws for the aggregate 'forces' due to deformation

$$\mathbf{M}^{T}{}_{u} \equiv - \int_V \partial_j \boldsymbol{\nu}_{i\,u}\, \sigma_{i\,j}\, dV$$

are conceptually much more demanding.

The reason is that the momentum productions in the body of matter are closely related to the first law of thermodynamics. This does not come as surprise if it is recalled that the balance of aggregate momentum is nothing else but the partial energy balances with respect to the non-holonomic speed.

Thus Prager's remark concerning the limitation of considerations to mechanical aspects, very often acceptable and adopted here as well, does not apply (1961/85):

> "*Wenn das Gegenteil nicht ausdrücklich festgestellt wird, sollen daher im Folgenden die Wechselwirkungen zwischen mechanischen und thermischen Vorgängen unberücksichtigt bleiben.* Mathematisch kommt das darauf hinaus, daß man nur die Kontinuitätsgleichung und die Bewegungsgleichung, *aber nicht die Energiegleichung benutzt …* " *Italics: MS.*

In aggregate mechanics the equation of motion, the momentum balance *is* the system of partial energy balances and accordingly the concepts of 'force' in elementary and in aggregate mechanics are different, 'by definition'.

19.3.2.4 FUNDAMENTAL COMPONENTS

The basic idea in setting up laws for the 'total stress force' in question is to conceive it axiomatically as the sum of a 'dissipative force' and a 'storage force'

$$\mathbf{M}^{T}{}_{u} = \mathbf{M}^{R}{}_{u} + \mathbf{M}^{S}{}_{u}\ ,$$

or, more in line with the usage in thermodynamics, as the sum of an 'irreversible dissipation of aggregate momentum' and of a 'reversible storage of aggregate momentum'.

Both these forces are aggregate momentum productions taking place *inside the body*. But different from the momentum production proper due to *external causes* dissipation and storage of aggregate momentum are due to *internal causes*. No wonder that the terminology is not always satisfactory and resulting in 'considerable' confusion. That it is possible to talk about 'dissipation of momentum' in a meaningful way is due to the fact that the balance of aggregate momentum is the system of the partial energy balances.

In the section on 'Potential energy, strain energy' following the discussion of special cases Eringen states (1962/116):

> "Consider now the general case of global energy conservation. *It is reasonable to assume that the stress tensor is divisible into two parts* … of which one is the reversible part of the stress, which will be called the hyper-elastic stress and the other part is the irreversible or dissipative part." *Italics: MS; italics in the reference ignored.*

This is an extremely vague way of talking about this fundamental matter. In a subsequent chapter Eringen discusses the theory of constitutive laws in general and for isotropic hyper-elastic materials and Stokesian fluids in particular (1962/132-171).

Concerning general constitutive laws Eringen warns (1962/ 135):

> " … It is not possible to develop general constitutive relations which encompass all possible special situations and materials, although some fairly general classes of materials may be brought under one set of constitutive equations. Such wide generalizations, however, often cloud the fundamental physical and mathematical ideas without making any essential contribution. Unifications and generalizations carry great temptations which must be overcome for the purpose of clarity in the fundamental ideas. We shall therefore pursue the modest approach … "

Contrary to Eringen's opinion the author has shown in many particular instances that the more general laws permit to grasp the fundamental ideas more clearly and apply them much more efficiently.

But after some speculative diversions the author fully agrees with Eringen's warning concerning general stress laws. In view of the intricacies of the kinematics of continua and of the constitutive laws in any particular case the appropriate stress law has to be invoked.

19.3.3 Example: Stokesian fluids

The following discussion of aggregation in Stokesian fluids is an attempt to provide some insights into the fundamentals, to demonstrate the problems encountered and to lead to an axiomatic approach, providing the sound basis of the current engineering practice.

The stress law for Stokesian fluids has been stated before. According to the Cayley-Hamilton[WR] theorem of matrix algebra the most general isotropic polynomial stress law to be derived is

$$\sigma_{ij} = -p\,\delta_{ij} + \upsilon_0\,\delta_{ij} + \upsilon_1\,v^{def}{}_{ij} + \upsilon_2\,v^{def}{}_{ik}\,v^{def}{}_{kj}\,,$$

where the viscosities

$$\upsilon_r = \upsilon_r\,(V_1, V_2, V_3)\,, \quad r = 0, 1, 2 \qquad \upsilon_0\,(0, 0, 0) = 0$$

are scalar functions of the scalar invariants of the deformation rate

$$v^{def}{}_{ij} \equiv (\partial_i v_j + \partial_j v_i)\,/2\,.$$

Further, remembering that only forcibly driven systems are under consideration the stress law becomes

$$\sigma_{ij} = = -p\,\delta_{ij} + \upsilon_0\,\delta_{ij} + \upsilon_1\,(\partial_i \mathbf{v}_{ju} + \partial_j \mathbf{v}_{iu})\,/2\,\mathbf{v}_u$$
$$+ \upsilon_2\,(\partial_i \mathbf{v}_{ku} + \partial_k \mathbf{v}_{iu})\,/2\,(\partial_k \mathbf{v}_{jv} + \partial_j \mathbf{v}_{kv})\,/2\,\mathbf{v}_u\,\mathbf{v}_v\,.$$

While for Newtonian fluids it is readily seen that potential flows are possible, an example having been discussed, the same has not been proved in the more general case of Stokesian fluids. The author admits not to have followed up this question.

Thus, and in view of the omnipresent turbulent boundary layers the following considerations are of a heuristic nature, and this even more so in view of the viscosities depending on the invariants of the total local deformation rate. In the following it will be shown how this problem can be overcome.

19.3.3.1 Reversible, irreversible components

Introducing the stress law into the definition of the aggregate 'force' due to deformation results in the explicit law for the aggregate 'Stokesian force'

$$\mathbf{M}^T{}_u = -\int_V \partial_j \mathbf{v}_{iu}\,[-p\,\delta_{ij} + \upsilon_0\,\delta_{ij} + \upsilon_1\,(\partial_i \mathbf{v}_{ju} + \partial_j \mathbf{v}_{iu})\,/2\,\mathbf{v}_u$$
$$+ \upsilon_2\,(\partial_i \mathbf{v}_{ku} + \partial_k \mathbf{v}_{iu})\,/2\,(\partial_k \mathbf{v}_{ju} + \partial_j \mathbf{v}_{kv})\,/2\,\mathbf{v}_u\,\mathbf{v}_v]\,dV\,.$$

In terms of components this law permits to

$$\mathbf{M}^T{}_u = \int_V \partial_i \mathbf{v}_{iu}\,p\,dV - \int_V \partial_i \mathbf{v}_{iu}\,\upsilon_0\,dV$$
$$- \int_V \partial_i \mathbf{v}_{iu}\,\upsilon_1\,(\partial_i \mathbf{v}_{uj} + \partial_j \mathbf{v}_{ui})\,/2\,dV\,\mathbf{v}_v$$
$$- \int_V \partial_i \mathbf{v}_{iu}\,\upsilon_2\,(\partial_i \mathbf{v}_{ku} + \partial_k \mathbf{v}_{iu})\,/2$$
$$(\partial_k \mathbf{v}_{jv} + \partial_j \mathbf{v}_{kv})\,/2\,dV\,\mathbf{v}_v\,\mathbf{v}_w\,.$$

The first term

$$\mathbf{M}^{S}{}_{u} = \int_V \partial_i \mathbf{v}_{i\,u}\, p \, dV$$

represents the reversible 'work' component of the first law of thermodynamics, while the remaining three 'resistance' terms

$$\mathbf{M}^{R}{}_{u} = \mathbf{M}^{R}{}_{u\,0} + \mathbf{M}^{R}{}_{u\,1} + \mathbf{M}^{R}{}_{u\,2}$$

$$\equiv - \mathbf{R}_{u} - \mathbf{R}_{u\,v}\, \mathbf{v}_{v} - \mathbf{R}_{u\,v\,w}\, \mathbf{v}_{v}\, \mathbf{v}_{w}$$

represent dissipative components with the aggregate 'dampings'

$$\mathbf{R}_{u} \equiv \int_V \partial_i \mathbf{v}_{i\,u}\, \upsilon_0 \, dV \equiv \mathbf{M}^{R}{}_{u\,0}$$

$$\mathbf{R}_{u\,v} \equiv \int_V \partial_i \mathbf{v}_{i\,u}\, \upsilon_1 \,(\partial_i \mathbf{v}_{j\,u} + \partial_j \mathbf{v}_{i\,u})/2 \, dV$$

$$\mathbf{R}_{u\,v} \equiv \int_V \partial_i \mathbf{v}_{i\,u}\, \upsilon_2 \,(\partial_i \mathbf{v}_{k\,u} + \partial_k \mathbf{v}_{i\,u})/2 \,(\partial_k \mathbf{v}_{j\,v} + \partial_j \mathbf{v}_{k\,v})/2 \, dV \, .$$

These formal definitions have been derived in detail to demonstrate explicitly that the 'dampings' introduced are by no means aggregate 'properties' to be determined accordingly, even if the viscosities are assumed to be independent of the invariants of the rate of deformation.

The situation is exactly the same as in case of the constitutive laws themselves as has been pointed out repeatedly. In general 'statistical' mechanics does not permit to determine the phenomenological parameters of the constitutive laws (Eringen, 1962/134).

19.3.3.2 AXIOMATIC SYSTEM MODEL BASED

The facts that most systems are not simple forcibly driven systems of the type treated before and that the difficulties to determine the aggregate forces along the route outlined are insurmountable leads directly to an axiomatic approach on the aggregate level exactly corresponding to the axiomatic approach in thermodynamics and in continuum mechanics.

Conveniently the balance of aggregate momentum for simple systems

$$d_t\,(\mathbf{I}_{u\,v}\, \mathbf{v}_{v}) + \mathbf{e}_{u\,v\,r}\, \mathbf{v}_{v}\, \mathbf{I}_{r\,w}\, \mathbf{v}_{w} = \mathbf{M}^{C}{}_{u} + \mathbf{M}^{M}{}_{u} + \mathbf{M}^{T}{}_{u} + \mathbf{M}^{P}{}_{u}$$

serves as the model of the axiomatic system and the laws for the different aggregate 'forces' are introduced as before, except for the aggregate 'forces' due to deformation.

The laws for the latter are introduced axiomatically and the phenomenological parameters, the aggregate 'properties' have in any case to be identified from physical experiments using prototypes, maybe on model scale, or computational experiments using numerical models.

19.3.3.3 AXIOMATIC 'RSISTANCE' LAW

In the case of great interest the body of matter, here in particular the fluid environment of a system of moving bodies, is incompressible and thus the reversible storage of aggregate momentum vanishes

$$\mathbf{M}^{S}{}_{u} = \mathbf{0}_{u} \,,$$

while the irreversible dissipation of aggregate momentum, the resistance 'force'

$$\mathbf{M}^{R}{}_{u} = - \mathbf{R}_{u\,v}\, \mathbf{v}_{v}$$

with the symmetrical, positive definite aggregate damping

$$\mathbf{R}_{u\,v} = \mathbf{R}_{v\,u}$$

is of great importance.

A rather simple minded approximate way to meet the requirement of positive definite damping is to consider the aggregate damping as generalised tensor polynomial of the state of motion. The latter may be described tentatively (or rather speculatively) by the dyadic product of the intensity and the extensity of motion

$$\mathbf{R}_{u\,v} = \Sigma_{p}\, \mathbf{R}_{u\,w\,p}\, (\mathbf{v}_{w}\, \mathbf{M}_{v} /2)^{p} = \Sigma_{p}\, \mathbf{R}_{u\,w\,p}\, (\mathbf{v}_{w}\, \mathbf{I}_{v\,r}\, \mathbf{v}_{r} /2)^{p} \,,$$

with the number of terms limited by the number of degrees of freedom

$$p = 0, \ldots, f - 1$$

due to the Cayley-Hamilton theorem already mentioned in the context of constitutive stress laws (Zurmühl, 1961/176-179).

Thus, if the aggregate damping components are positive definite the aggregate damping is positive definite as required. The reason for introducing the momentum at this stage is to account for non-symmetric boundary conditions as far as possible.

The remaining symmetry is not realistic in case of speed reversal relative to the asymmetries. Apart of the 'inherent' symmetry there maybe additional symmetries due to the boundary conditions (Schmiechen, 1970.m).

Thus, the latter 'result' does not put an end to the discussions concerning the appropriate approximation the 'forces' in question. Readers will remember that the gyroscopic, powerless term, the internal convective flow of aggregate momentum is of second order in the generalised speed.

19.3.3.4 APPROXIMATE RESISTANCE LAWS

In practice it will often be sufficient to consider a reduced number of terms with constant aggregate dampings. Of great practical interest are the zeroth order and the first order approximations

$$\mathbf{R}^{0}{}_{u\,v} = \mathbf{R}_{u\,v\,0}\ .$$

$$\mathbf{R}^{1}{}_{u\,v} = \mathbf{R}_{u\,v\,0} + \mathbf{R}_{u\,w\,p}\,\mathbf{v}_{w}\,\mathbf{I}_{v\,r}\,\mathbf{v}_{r}/2\ ,$$

$$= \mathbf{R}_{u\,v\,0} + \mathbf{R}_{u\,w\,1}\,\mathbf{v}_{w}\,\mathbf{I}_{v\,r}\,\mathbf{v}_{r}/2\ ,$$

respectively.

The resistance laws corresponding to the approximate damping laws are in zeroth and in first order

$$\mathbf{M}^{R\,0}{}_{u} = \mathbf{R}_{u\,v\,0}\,\mathbf{v}_{v}$$

$$\mathbf{M}^{R\,1}{}_{u} = \mathbf{R}_{u\,v\,0} + \mathbf{R}_{u\,w\,1}\,\mathbf{v}_{w}\,\mathbf{I}_{v\,r}\,\mathbf{v}_{r}\,\mathbf{v}_{v}/2\ ,$$

$$= -\,(\mathbf{R}_{u\,v\,0} + \mathbf{R}_{u\,v\,1}\,E)\,\mathbf{v}_{v}\ ,$$

respectively.

The latter format has been used in a radically reduced format in case of a dominant speed component replacing the norm of the speed with its magnitude

$$|\,\mathbf{v}\,| = (\mathbf{v}_{w}\,\mathbf{v}_{w})^{1/2}\ .$$

(Schmiechen, 1962). In any case the damping is not an invariant 'property'.

19.3.3.5 DISSIPATION FUNCTIONS

Thus the patent idea to derive the aggregate resistance 'force' from the power of dissipation

$$\mathbf{M}^{R}{}_{u} = -\,\partial\,P^{R}/\partial\,\mathbf{v}_{u}\ ,$$

as the aggregate momentum itself

$$\mathbf{M}_{u} = \partial\,E/\partial\,\mathbf{v}_{u} = \mathbf{I}_{u\,v}\,\mathbf{v}_{v}$$

is derived from the kinetic energy, works only in zeroth approximation. The latter case plays a prominent role in linear elasto-statics.

As far as the author has scanned the literature he has found the subject of dissipation and dissipation functions in aggregate mechanics treated unsatisfactorily. Lurie provides three sections on this matter, the last section on 'Aerodynamic resisting force' concerned with the ballistics of spinning shells (2002/248-265).

Although not usually mentioned in textbooks the theory of moving bodies has not only been developed in view of planetary motions but always in view of military applications, motions of bodies under less ideal conditions.

Among these are not only spinning shells, torpedoes and ballistic missiles of recent times but arrows and boomerangs, the latter used doubtlessly since prehistoric times, in historical times by the ancient Egyptians, not only by the aborigines in Australia usually considered as 'inventors' of boomerangs. Today the ancient weapons are used in sport competitions as is the discus of the ancient Greeks.

19.3.3.6 RELATED TO DIRECT AGGREGATION

At this stage it will be recalled that the direct aggregation formally resulted in the law

$$\mathbf{M}^{R}{}_{u} = - \mathbf{R}_{u\,v}\,\mathbf{v}_{v} - \mathbf{R}_{u\,v\,w}\,\mathbf{v}_{v}\,\mathbf{v}_{w}$$

$$= - (\mathbf{R}_{u\,v} + \mathbf{R}_{u\,v\,w}\,\mathbf{v}_{w})\,\mathbf{v}_{v}\,,$$

with the constant term vanishing

$$\mathbf{R}_{u} = \mathbf{0}_{u}$$

due to appropriate choice of the generalised non-holonomic speed.

The difference between the two laws appears to be small but is profound. While in the approximate law stated before the damping is positive definite 'by definition', the same is not true in general of the law suggested by direct aggregation. The author remembers having tried very hard to come up with appropriate conditions to be imposed, the present proposal being a continuation of these speculations.

Similar problems arise in elasto-mechanics. Appropriately accounting for non-linearities of multi-component balances is by no means as simple as 'experts' belief. The problems arise if load reversals have not been taken care of already during calibrations (Schmiechen, 1980.c).

19.3.4 Example: hyper-elastic material

The other special case of great theoretical and engineering interest is the 'opposite' case of elastic bodies, the elementary theory being covered in pertinent textbooks in considerable detail necessary for practical applications.

In this case the reversible storage of aggregate momentum is of primary concern, while the irreversible dissipation of aggregate momentum is considered to be negligible. This case may be treated closely following the foregoing example of Stokesian fluids. Despite the great importance the author will not develop the details, as they do not contribute to the purpose of this treatise.

Only a few remarks are in order, but will be delayed to the end of the chapter on the special case of holonomic speed.

19.3.5 Euler-Lagrangean equation: explicit

19.3.5.1 COMPLETE STATE MODEL

With the laws of the aggregate forces the balance of aggregate momentum, the Euler-Lagrangean equation for the system of bodies becomes explicitly

$$d_t \mathbf{M}_u + e_{u\,v\,w} \mathbf{v}_v \mathbf{M}_w \mathbf{v}_w = \mathbf{M}^M{}_u + \mathbf{M}^R{}_u + \mathbf{M}^S{}_u + \mathbf{M}^P{}_u \,,$$

the aggregate momentum, the extensity of motion

$$\mathbf{M}_u = \mathbf{I}_{u\,v} \mathbf{v}_v \,,$$

corresponding to the generalised speed, the intensity of motion

$$\mathbf{v}_v = = \mathbf{I}^{-1}{}_{v\,u} \mathbf{M}_u$$

and all forces depending on the generalised position, which in turn is subject to the kinematical state equation

$$d_t \mathbf{s}_u = \mathbf{K}_{u\,v} \mathbf{v}_v \,.$$

Discussions of this state space model in terms of non-holonomic speed suggest that the essence has not 'really' been 'understood', in Goethe's sense (Lurie, 2002/526). Lack of intuitive terminology and of corresponding operational notation further adds to the difficulties in following expositions and derivations.

Hamel ends the derivation of the 'equations of motions' together with the kinematical state equation with the succinct sentence: 'Sie haben allgemeine Gültigkeit.' followed by the short section '237. Warnung und Bemerkung' (Hamel, 1949/480-482). The present author stresses: *The state space model is valid only under the assumptions made.*

Talking in terms of Christoffel 'symbols'(/309-312), 'three index symbols' (/394) instead of generalised operators and repeatedly resorting to symbolic notation for the exterior vector products in case of rigid body motions (/436 ff) does not support ready implementation in computer codes.

19.3.5.2 SIMULATION

The explicit solution for the 'acceleration'

$$\mathrm{d}_t \mathbf{v}_s + \mathbf{d}_{s\,v\,w}\, \mathbf{v}_v\, \mathbf{v}_w = \mathbf{I}^{-1}{}_{s\,u}\, [\, -\, (\mathrm{d}^{v}{}_{t}\, \mathbf{I}_{u\,v})\, \mathbf{v}_v + \mathbf{M}^{M}{}_{u} + \mathbf{M}^{R}{}_{u} + \mathbf{M}^{S}{}_{u} + \mathbf{M}^{P}{}_{u}].$$

with the inertia specific generalised rotor operator

$$\mathbf{d}_{s\,v\,w} \equiv \mathbf{I}^{-1}{}_{s\,u}\, \mathbf{e}_{u\,v\,r}\, \mathbf{I}_{r\,w}$$

traditionally called Christoffel 'symbols' of the second kind (Lurie, 2002/311 f.).

In general the inertia of systems is not invariant and thus the evaluation of the balance is not particularly efficient. If instead of the speed the aggregate momentum, the extensity of motion, *quantitas motus*, is treated as state, the state space model and the corresponding computer codes remain 'completely' transparent. Even in case of invariant inertia the aggregate momentum remains the preferred choice of state.

Incidentally this 'natural' choice of states, momentum and position, adopted in this treatise from the 'beginning', starting with the exposition of meta- mechanics *ad hoc*, corresponds to the states in the formulation Hamilton's principle as will be shown.

The format of the momentum balance solved for the acceleration is 'standard' not only in 'analytical' mechanics, but also in the theory of general relativity and in quantum mechanics, though without most of the 'forces' of interest in classical mechanics (Wikipedia):

> "The Christoffel symbols find frequent use in Einstein's theory of general relativity, where spacetime is represented by a curved 4-dimensional Lorentz manifold with a Levi-Civita connection."

Concerning applications in quantum mechanics the following randomly selected abstract illustrates the point in question (DeWitt, 1952):

> "An isomorphism is shown to exist between the group of point transformations in classical mechanics and a certain subgroup of the group of all unitary transformations in quantum mechanics. This isomorphism is used to indicate that the quantum analogs of physically significant classical expressions can be constructed uniquely in any coordinate system. There is no ambiguity in the ordering of non-commuting quantum operators, and the method of constructing the quantum analogs is covariant under general coordinate transformations. *The method is actually only applicable to systems having Lagrangians which are at most quadratic in the velocities, but this includes all systems which are presently of interest in physics.* The method is applied to two intrinsically nonlinear examples, one of which is the gravitational field. The correct Hamiltonian operator for a quantized version of Einstein's gravitational theory is constructed." *Italics: MS.*

19.3.5.3 SINGLE BODY UNIVERSE

If a 'universe' is being considered consisting of a single body of matter in infinite space the generalised equation of motion reduces to

$$d_t \mathbf{M}_u + e_{u\,v\,w} \mathbf{v}_v \mathbf{M}_w = \mathbf{M}^R{}_u + \mathbf{M}^S{}_u + \mathbf{M}^P{}_u$$

accounting for the fact that aggregate diffusive momentum flows from 'outside' do not occur

$$\mathbf{M}^M{}_u = \mathbf{0}_u$$

the 'universe' considered has no environment 'providing' for such flows.

In general, the configuration of a 'universe', its mass distribution and thus the generalised tensor of its aggregate inertia is not invariant with respect to the speed space.

As has been observed earlier in the preferred observation spaces the total momentum production vanishes

$$\mathbf{M}^P{}_i = \int_m f_i \, dm = 0_i \,.$$

This does not imply that the aggregate momentum production vanishes as well.

Further, talking about a 'universe' we cannot refer to frames outside the 'universe'. Thus, as for rigid bodies, the positions of the bodies themselves are the reference 'frames', 'Bezugsmollusken' as Einstein called them. The conceptual problems are beyond the horizon of the present treatise.

EVALUATIONS/ASSESSMENTS

The special cases considered, the assumptions of steady states, of aggregate momentum productions 'derived' from energy 'potentials' and of aggregate diffusive flows of momentum 'derived' from a dissipation functions, are constitutive equations providing classes of powerful models. In practice position spaces with only few degrees of freedom often permit a rather precise description of the motions of mechanical systems.

CONCLUSIONS

To continue calling these special cases 'principles of mechanics' is more than misleading, nourishing expectations not warranted. But this usage of the seventeenth and eighteenth century is 'consistent' with the poor standard of presentation of classical mechanics in general.

19.4 'Degenerate' case: holonomic speed

PROBLEMS

At this stage the question arises how the famous 'variational principles of mechanics' are related to the weighted integrals of the local momentum balance, of Cauchy's universal equation of motion of continua.

MODELS

Following the derivation of the partial energy balances for non-holonomic generalised speed the Lagrangean equation may be arrived at as special case for holonomic generalised speed, the instance with degenerate kinematics and further restrictive assumptions.

GOALS/PLANS

The goal is to show explicitly the well known fact that the 'principles of mechanics' are all equivalent ramifications of the Lagrangean equation subject to restrictive assumptions of only limited interest in classical mechanics as practiced by scientists and engineers.

19.4.1 Lagrangean equation

19.4.1.1 GENERAL EQUATION

In the special case of holonomic generalised speed, in case of degenerate kinematics

$$\mathbf{K}_{v\,u} = \mathbf{U}_{v\,u}$$

the balance of aggregate momentum, the generalised Lagrangean equation reduces to the Lagrangean equation 'proper'

$$d_t\,(\boldsymbol{\partial}^{\mathbf{v}}{}_u\,E) - \boldsymbol{\partial}^{\mathbf{s}}{}_u\,E = \mathbf{M}^{M}{}_u + \mathbf{M}^{T}{}_u + \mathbf{M}^{P}{}_u\,.$$

Due to the relation

$$\boldsymbol{\partial}^{\mathbf{s}\,\mathbf{v}}{}_u = \boldsymbol{\partial}^{\mathbf{s}}{}_u$$

its 'format' does not differ from that of the Euler-Lagrangean equation for non-holonomic generalised speed

$$\partial_t\,\mathbf{M}_u + \mathbf{e}_{u\,v\,w}\,\mathbf{v}_v\,\mathbf{M}_w = \mathbf{M}^{M}{}_u + \mathbf{M}^{T}{}_u + \mathbf{M}^{P}{}_u$$

with the momentum

$$\mathbf{M}_u = \mathbf{I}_{u\,v}\,(\mathbf{s})\,\mathbf{v}_v$$

and the degenerate kinematical state equation

$$d_t\,\mathbf{s}_u = \mathbf{v}_u\,,$$

implying that the spaces of generalised position and generalised speed coincide.

This simple 'instantiation' completes the 'derivation' of the Lagrangean equation. No wonder that in his 'Dynamics' limited to the consideration of simple non-holonomic systems Kane[TR] hardly mentions the Lagrangean equation (1985/51), but leaves it as a students' exercise including the Passerello-Huston equation for non-holonomic systems (1985/327 f).

19.4.1.2 Very special case

In 'analytical' mechanics usually only systems of rigid bodies

$$\mathbf{M}^{T}{}_{u} = \mathbf{0}_{u}$$

in vacuo

$$\mathbf{M}^{M}{}_{u} = \mathbf{0}_{u}$$

and potential momentum production

$$\mathbf{M}^{P}{}_{u} = -\,\partial^{s}{}_{u}\,E^{P}$$

are being considered.

This ideal case

$$d_t\,(\partial^{v}{}_{u}\,E) - \partial^{s}{}_{u}\,E = -\,\partial^{s}{}_{u}\,E^{P},$$

usually written in the format

$$d_t\,(\partial^{v}{}_{u}\,E^{L}) - \partial^{s}{}_{u}\,E^{L} = \mathbf{0}_{u}$$

with the Lagrangean potential

$$E^{L} \equiv E - E^{P}$$

was of great interest in the eighteenth century and still is in celestial mechanics and in quantum mechanics.

19.4.1.3 Slightly generalised

The Lagrangean equation may be generalised to include non-rigid bodies if the aggregate force due to deformation of the body may be derived from an elastic potential

$$\mathbf{M}^{S}{}_{u} = -\,\partial^{s}{}_{u}\,E^{S}\,.$$

In that case the Lagrangean potential may be replaced by the more general potential

$$E^{L} \equiv E - E^{S} - E^{P}\,.$$

This case of great importance in engineering, in elasto-statics in particular.

19.4.1.4 Assessment revisited

Concerning the importance of these equations there has never been any doubt (Szabó, 1987/130):

> "Hinsichtlich der Anwendungen sind diese (Differential-)Gleichungen die wichtigsten Ergebnisse der Lagrangeschen Mechanik. Diesen Tatbestand scheint selbst Lagrange durch die Titelgebung 'Differentialgleichungen für die Lösung aller Probleme der Dynamik' zum Ausdruck bringen zu wollen."

Today the promising title 'Differential equations for the solution of all problems of dynamics' of the crucial section in Lagrange's 'Méchanique analytic', Part II, Section IV, needs to be supplemented by the qualification '*all problems of dynamics considered at the time of Lagrange*' (1997/223-236).

At present they are only of limited interest in classical mechanics as they do not even permit to treat rigid body motions, the simplest case of non-holonomic generalised speed, forgetting about diffusion and dissipation of aggregate momentum.

In most expositions, considering only holonomic generalised speed, the beautiful theory of Lagrange developed, maybe to impress students, is soon forgotten and Newton's and Euler's first principles are resorted to when discussing the motions of rigid bodies, if at all.

Accordingly the term 'Newton-Euler equations' appears to have become standard jargon in German lecture notes and textbooks (Kreuzer[E], 2006; Gasch, 1987). The author came across this term only while finishing this manuscript and had no chance to trace its origin.

Surprisingly Lurie discusses rigid body kinematics and dynamics before developing the general theory of aggregate mechanics. This may have been adequate when systems of rigid bodies were of primary interest. In the context of the present treatise simple bodies of matter, not limited to six or rather twelve degrees of freedom, are considered as 'building blocks' of more intricate systems.

19.4.2 Hamilton's 'canonical' equations

> "War es ein Gott, der diese Zeichen schrieb ... ".
>
> *Johann Wolfgang Goethe: Faust I. (BA 08/163). Motto of Lanczos's chapter on 'The Canonical Equations of Motion' (1986/161).*

19.4.2.1

Any state space model may be transformed into equivalent forms convenient for the purpose at hand (Wymore, 1967; Schmiechen, 1973). The instances to be considered are the 'canonical' transformations of the mechanical state space model in case of 'forces' due to potentials

$$d_t (\mathbf{I}_{u\,v}\, \mathbf{v}_v) = -\,\partial^s{}_u (E^S + E^P)$$

$$d_t\, \mathbf{s}_u = \mathbf{v}_u$$

with the speed and the position as components of the state, the two state 'variables'. The transformation of interest is the Legendre transformation, amounting to the exchange of the 'variables', the arguments. As usual the transformation is assumed to be one-to one (ein-eindeutig) (Lanczos 1986/161-167).

In the state model to be transformed the aggregate momentum

$$\mathbf{I}_{u\,v}\, \mathbf{v}_v = \mathbf{M}_u = -\,\partial^s{}_v\, E$$

represents the extensity of motion, while the generalised speed represents the intensity of motion.

19.4.2.2 LEGENDRE TRANSFORMATION

The Legendre transformation permits to transform a function f (x, y) into a another function F (X, Y), the 'new' arguments defined by the total differential

$$d\, f(x, y) = X\, dx + Y\, dy$$

or explicitly

$$X \equiv \partial f(x, y)/\partial x \equiv \partial^x f(x, y)$$

and

$$Y \equiv \partial f(x, y)/\partial y \equiv \partial^y f(x, y)\,.$$

If the 'new' function is defined by

$$F(X, Y) = x\, X + y\, Y - f(x, y)$$

the total differential becomes

$$d\, F(X, Y) = x\, dX + y\, dY$$

and the following reciprocal relations hold

$$x \equiv \partial F(X, Y)/\partial X \equiv \partial^X F(X, Y)$$

$$y \equiv \partial F(X, Y)/\partial Y \equiv \partial^Y F(X, Y)\,.$$

19.4.2.3 HAMILTON'S EQUATIONS

Accordingly the original state space model is transformed into Hamilton[WR]'s so called canonical equations

$$d_t\,\mathbf{M}_u = -\,\partial\,E^H/\partial\,\mathbf{s}_u\,,$$

$$d_t\,\mathbf{s}_u = \partial\,E^H/\partial\,\mathbf{M}_u\,,$$

if the Hamiltonian energy potential, that is the total energy

$$E^H \equiv \mathbf{M}_u\,\mathbf{v}_u - E^L = 2\,E - (E - E^S - E^P) = E + E^S + E^P$$

is being introduced with the definitions of the aggregate momentum

$$\mathbf{M}_u \equiv \partial\,E/\partial\,\mathbf{v}_u = \partial\,E^H/\partial\,\mathbf{v}_u\,,$$

of the aggregate force

$$\mathbf{F}_u \equiv \partial\,E/\partial\,\mathbf{s}_u = \partial\,E^H/\partial\,\mathbf{s}_u$$

and of the generalised speed

$$\mathbf{v}_u \equiv \partial\,E/\partial\,\mathbf{M}_u = \partial\,E^H/\partial\,\mathbf{M}_u\,.$$

The result shows that the new, so called canonical components of state are the aggregate momentum and the generalised position. So only the intensity of motion has been exchanged for the extensity of motion. Though the beauty and the intuitive appeal of these equations are overwhelming there is clearly nothing 'really divine' about them.

In the context of the present exposition there is even nothing 'new' about them. As has already been mentioned aggregate momentum and generalised position are the 'natural' state components used throughout this treatise.

19.4.2.4 RECEPTION OF HAMILTON'S EQUATIONS

In an appendix Lanczos explicitly shows that (1986/397 ff):

> "The important transition from Lagrangean to the Hamiltonian form of dynamics can be accomplished in a more direct way, without the use of the Legendre transformation, basing the argument solely on the method of the Lagrangean multiplier."

Concerning the reception of Hamilton's equations Arnold notes in the preface of his 'Mathematical Methods of Classical Mechanics' (1978/IX):

> "Characterizing analytical dynamics in his 'Lectures on the development of mathematics in the nineteenth century' Felix Klein wrote that ' … a physicist for his problems can extract from these theories only little and engineers nothing.' The development of the sciences in the following years decisively disproved this remark. Hamiltonian formalism … has become one of the most often used tools in the mathematical arsenal of physics. …,

Hamilton's equations began to be used constantly in engineering calculations.'

To put things into proportion it needs to be added that engineers can use these equations only if the model is appropriate, if the generalised speed is holonomic and if the aggregate forces are potential and if there is neither momentum diffusion nor dissipation, all conditions being rather the exception than the rule in engineering problems.

Sommerfeld discusses Hamilton[WR]'s theory in some detail (1953/208-230) and the Legendre transformation in general at the end of a chapter on Routh's equations and cyclical systems, 'wir haben sie hier hauptsächlich deshalb zur Sprache gebracht, um daran in der Thermodynamik erinnern zu können' (1953/214-217).

The discussion has shown that the theory of state space models is prerequisite for an adequate understanding of the theory. This theory is badly felt missing in most expositions or only vaguely sketched *ad hoc* even in advanced textbooks (Lurie, 2002/526-530).

In order to spread the gospel the author has held lectures on 'Platons Höhlen-Gleichnis und die Theorie der Zustands-Modelle' at the University of Rostock and the Technical University Hamburg-Harburg, the handouts to be found on his website.

19.4.3 Further ramifications?

In his 'Thermodynamik' Sommerfeld devotes a chapter on these transformations under the heading 'Die thermodynamischen Potentiale und die Reziprozitäsbeziehungen' including an instructive 'matrix' of the relations for four thermodynamical potentials (1965/33-38).

Subsequently the question arises whether further canonical transformations of the mechanical potentials are possible including an exchange of the generalised position for another component of state. On closer inspection such transformations are not necessary and not possible in case of holonomic generalised speed.

But in case of non-holonomic generalised speed the generalised tensor of kinematics has been introduced as 'integrating factor' corresponding to temperature in thermodynamics. In that case the original state space model is

$$d_t\,(\mathbf{I}_{u\,v}\,\mathbf{v}_v) = \mathbf{K}_{v\,u}\,\partial^s{}_v\,E$$

$$d_t\,\mathbf{s}_u = \mathbf{K}_{u\,v}\,\mathbf{v}_v\,.$$

Thus the speed in the position space

$$\mathbf{V}_u \equiv \mathbf{K}_{u\,v}\,\mathbf{v}_v$$

may be introduced as state and the corresponding potential

$$E^V \equiv \mathbf{M}_u\,\mathbf{V}_u - E^L = \mathbf{M}_u\,\mathbf{V}_u - E + E^P .$$

Accordingly the state space model assumes the format

$$d_t\,\mathbf{M}_u = \partial^v{}_u\,\mathbf{V}_v\,\partial^s{}_v\,E$$

$$d_t\,\mathbf{s}_u = \mathbf{V}_u .$$

Mixed state space models are obtained if only part of the intensity of motion is exchanged for the extensity of motion. Sommerfeld discusses the case of Routh's potential and equations. The author will not further follow these ramifications, leading away from the goal of this treatise, but considers them as students' exercises.

19.4.4 Absent, cyclic speeds

At times when computers where not available the solution of the canonical equations for realistic systems posed insurmountable difficulties. Thus generalised speeds not occurring in the power functions were of particular interest.

They have first been discussed by Routh (1877) as 'absent' and by Helmholtz (1884) as 'cyclic'. In the literature the 'absent' velocities are also found under the names 'kinosthenic', due to Thomson, and 'ignorable', due to Whittaker (Lanczos 1986/125). Päsler (1968) devotes a chapter to cyclic speeds without mentioning Hertz.

In any case of an absent generalised speed

$$d_t\,\mathbf{M}_p = -\,\partial\,E^H/\partial\,\mathbf{v}_p = \mathbf{0}_p$$

the corresponding aggregate momentum is constant

$$\mathbf{M}_p = \mathbf{c}_p = \mathbf{const}_p .$$

Accordingly problems can be simplified if the number of absent velocities can be increased by transformations, invariants of which are the Hamiltonian equations. The permissible transformations, the canonical transformations, are at the core of Jacobi's theory of integration. (Lanczos 1986/193 ff). Päsler notes (1968/115), that for cyclic velocities Routh's mixed equations, partly Lagrangean and partly Hamiltonian, offer particular advantages.

Absent speeds can be eliminated before a problem is solved and they can later be obtained by 'quadrature', by integration. The elimination leads to apparent potentials, which gave rise to Hertz' speculations mentioned before (Lanczos 1986/130 ff).

Today the solution of intricate differential equations is by no means trivial, but in many cases easily possible. Accordingly Kane[TR] states (1985/X):

> "Now that computers enable one to extract highly valuable information from large sets of complicated equations of motion, all this has changed. In fact, the inability to *formulate* equations of motion effectively can be as great a hindrance at present as the inability to *solve* equations of motion was formerly."

19.4.5 Elasto-mechanics: continued

In zeroth approximation the force is assumed to be derived from a potential

$$\mathbf{M}^{S}{}_{u} = -\,\partial\, P^{S}/\partial\, \mathbf{s}_{u}\ .$$

Consequently the balance of aggregate momentum assumes the format

$$d_{t}\,(\partial\, E/\partial\, \mathbf{v}_{u}) - \partial\,(E - E^{S} - E^{P})/\partial\, \mathbf{s}_{u} = \mathbf{0}_{u}$$

and with the Lagrangean potential

$$E^{L} \equiv E - E^{S} - E^{P}$$

as before in the format of the Lagrangean equation in Hamilton[WR]'s format

$$d_{t}\,(\partial\, E^{L}/\partial\, \mathbf{v}_{u}) - \partial\, E^{L}/\partial\, \mathbf{s}_{u} = \mathbf{0}_{u}\ .$$

In the static case this equation reduces to

$$\partial\,(E^{S} + E^{P})/\partial\, \mathbf{s}_{u} = \mathbf{0}_{u}\ .$$

Szabó's exposition of the subject starts with 'Elementary problems' in 'Einführung in die Technische Mechanik' (1984/83-196) and continues in 'Höhere Technische Mechanik' with 'Prinzipien der Mechanik' (1956/1-132) before 'Ausgewählte Probleme der höheren Elastizitätstheorie are treated (1956/132-319).

In Gummert's advanced 'Mechanik', *expressis verbis* no longer 'Technische Mechanik' the exposition (1994/266-449) includes elementary 'energy methods' (1994/402-429) and under 'Prinzipien der Mechanik' a section on 'Statik deformierbarer Systeme' (1994/711-734).

Concerning general aspects Morgenstern's exposition appears less satisfactory (1961/69-148), but the proofs of approximate theories are of general interest (1961/120-128).

As in other cases the author feels that for beginners the basic ideas and general principles are still blurred by unnecessary artifacts and technical details 'later' necessary for applications. But the author admits never having taught a course on the subject and will not follow up the subject in the context of this treatise as has been stated before.

19.4.6 'Statistical' mechanics

"Die Natur sei inkommensurabel, und bei den großen Irregularitäten sei es sehr schwer, das Gesetzliche zu finden."

Johann Wolfgang Goethe: 22.03.1824 (Eckermann, 1911/78).

19.4.6.1 General theory

One of the prominent applications of Hamilton's equations is in 'statistical', correctly probabilistic mechanics, originally trying to exploit the concept of matter consisting of particles on the basis of general dynamics in terms of Hamilton's canonical equations

$$d_t \mathbf{M}_u = -\partial E^H / \partial \mathbf{s}_u ,$$

$$d_t \mathbf{s}_u = \partial E^H / \partial \mathbf{M}_u .$$

for a system with f degrees of freedom

$$u = 1, \dots f .$$

It is assumed that the Hamiltonian function

$$E^H = E^H (\mathbf{s}_u , \mathbf{M}_u) = \text{const}$$

is independent of time and that the number of degrees of freedom is very large

$$f \rightarrow \infty .$$

Further 'no assumptions are being made concerning the nature of the particles and of the forces acting between them in order to keep the results as general as possible' – under the limiting assumptions made before. Thus 'by definition' the theory cannot provide phenomenological parameters of stress laws, 'constitutive' laws of different materials and of any other 'physical' laws.

Talking about stress laws it is immediately noted that statistical theory cannot only serve as the basis of the kinetic theory of gases but of turbulence theory as well. In this case the particles are rather different from atoms and molecules. But if reference is made to the statistical theory of turbulence this theory has originally not been developed in the spirit of statistical mechanics to be discussed (Friedlander, 1961).

19.4.6.2 TWO EXAMPLES

Only two examples from the kinetic theory of gases may serve to illustrate the point. In order to explain the observed macroscopic behaviour the model of the ideal gas with particles of vanishing volumes is not sufficient for many practical applications. The 'nature', at least 'the' volume of the particles and 'the' cohesive pressure has to be taken into account, as has been done in van der Waals' gas law (Schmidt^{Ern}, 1953/206-213).

In the other example the specific heat depends on the degrees of freedom of the particles, thus the dependence on the number of atoms in the molecules (Schmidt^{Ern}, 1953/40-44). In view of the deviations from the results of the theory of ideal mono-atomic gases Chintschin notes (1964/147):

> "Vorhandene Abweichungen haben in der Mehrzahl der Fälle eine befriedigende physikalische Erklärung gefunden, die hauptsächlich darauf beruht, daß die Eigenschaften realer Gase von denen der idealen Gase abweichen."

This sentence is the wrong way round: the 'theoretical behaviour' deviates from the real behaviour.

The deviations are due to the fact that the kinetic theory purposely excludes certain aspects of the nature of the particles. In order to obtain meaningful results not all aspects can be ignored. Chintschin does not explicitly mention the degrees of freedom of the particles, which are 'essential' to the theory from the beginning. The energy is distributed uniformly over the degrees of freedom, not over the particles.

As the 'nature' of the forces between the particles remains unspecified Chintschin claims that statistical mechanics is not strictly 'mechanistic'. For the same reason classical mechanics as a whole is claimed not to be 'mechanistic'. No assumptions are made concerning the 'nature' of 'forces', the diffusive momentum flux and the momentum production, neither in local nor in global, aggregate mechanics.

The diffusive momentum flux has been ascribed to the molecular structure of matter but the 'nature' of forces between the molecules in fluid and solid bodies of matter remains 'open'. The momentum production has been ascribed to the nuclear structure of matter but the 'nature' of the forces between the constituents of the nucleons remains an 'open' question.

But even the molecular and the nuclear models of matter are not necessary, only the macroscopic laws are of interest in mechanics, else classical mechanics could not have been developed to its present state. The underlying continuum model permits much more general interpretations than that in terms of molecules as the theory of turbulence shows.

19.4.6.3 LIMITS OF THE THEORY

As has been shown in the development of the theory of aggregate mechanics even the stress laws are no longer sufficient to describe the aggregate behaviour of bodies of matter. As has been mentioned the situation is the same as in thermodynamics.

Contrary to the widely held opinion statistical mechanics does not provide for the foundation of thermodynamics but 'only' for 'some' explanations. Thermodynamics is a coherent macroscopic theory the axiomatic foundation of which has inspired the present exercise.

Chintschin (1964/18) notes that 'general dynamics alone, the basis of statistical mechanics, in large parts uses concepts and methods of the probability theory before the large, infinite number of particles comes in, the latter permitting to apply methods of probability theory to determine asymptotic equations'. Chintschin continues:

> "In den meisten Darstellungen werden diese Gleichungen ohne irgendwelche Rechtfertigung benutzt, wie wir schon vorher bemerkt haben. Die Autoren leiten ihre Gleichungen zunächst für irgendwelche besonders einfachen Spezialfälle ab (beispielsweise für den Fall eines einheitlichen einatomigen idealen Gases); ohne Begründung werden sie dann für den allgemeinen Fall übernommen, bestenfalls werden einige Argumente heuristischen Charakters angefügt. Die Methode von Fowler bildet die vielleicht einzige Ausnahme dieser allgemeinen Regel. Wie schon erwähnt, haben Darwin und Fowler eine Methode gefunden, um asymptotische Gleichungen zu erlangen; zu ihrer Begründung wurde dann ein sehr spezieller und reichlich umständlicher analytischer Apparat entwickelt. Die expliziten Resultate der Wahrscheinlichkeitstheorie wurden nirgendwo benutzt, die Autoren errichten statt dessen ein neues logisches Gebäude; sie bewegen sich aber in der Tat auf analytischen Bahnen, die denen der Wahrscheinlichkeitstheorie bei der Ableitung ihrer Grenzwertsätze parallel gehen. Es bleibt hier nur ein einziger Schritt übrig, um zu der Methode zu gelangen, die wir als die nützlichste betrachten dürfen: Statt in komplizierter Formulierung den ganzen langen analytischen Prozeß zu wiederholen, der zu den Grenzwertsätzen der Wahrscheinlichkeitstheorie führt, wollen wir versuchen, sogleich die Brücke zu betreten, die diese beiden Problemgruppen verbindet; wir wollen die Gleichung finden, die diesen Übergang vermittelt

> und somit das gesamte Asymptotenproblem der statistischen Mechanik auf die bekannten Grenzwertsätze der Wahrscheinlichkeitstheorie reduziert."

Since the early days of Birkhoff the theory of systems has been further developed. Noteworthy is 'A new approach to statistical mechanics' by Wiener (1966). Most dramatic is the 'progress' since the discovery of strange attractors and deterministic chaos (Leven, 1994). These developments open a new approach to turbulence although in the index of the monograph of Leven and co-authors the entry 'turbulence' does not occur!

19.4.6.4 CONCEPTUAL PROBLEMS

In view of the interaction of the particles the assumption of uniform distribution of energy among the degrees of freedom is causing serious conceptual problems (Chintschin, 1964/48 f):

> "Wir wollen diese kurzen einführenden Erörterungen mit dem Hinweis abschließen, daß die Zerlegung eines Systems in Komponenten zu einem spezifischen methodischen Paradoxon führt, wie dies schon oft bemerkt wurde. Wie wir schon am Anfang erwähnten, wird trotz der Allgemeinheit und Abstraktheit der Hypothesen der statistischen Mechanik doch stets angenommen, daß die Teilchen der Materie in einem Zustand intensiver energetischer Wechselwirkung sind, wobei die Energie eines Teilchens auf ein anderes übertragen wird (beispielsweise durch Stöße). Wie wir später noch genauer sehen werden, beruhen die Methoden der statistischen Mechanik genau auf dieser Möglichkeit des Energieaustausches zwischen verschiedenen Teilchen, die in der Materie enthalten sind. Wenn wir jedoch die Teilchen, die zu einem bestimmten physikalischen System gehören, als Komponenten in dem oben definierten Sinne auffassen, schließen wir die Möglichkeit energetischer Wechselwirkung zwischen ihnen aus. Wenn also die Hamilton-Funktion, die die Energie unseres Systems ausdrückt, eine Summe von Funktionen ist, die jeweils nur von den dynamischen Koordinaten eines einzigen Teilchens abhängen (und die Hamiltonsche Funktion dieses Teilchens darstellen), dann spaltet sich in der Tat das Gleichungssystem … in Komponenten auf; jede von ihnen beschreibt die Bewegung eines getrennten Teilchens und ist in keiner Weise mit den anderen Teilchen verknüpft. Somit ist die Energie eines einzelnen Teilchens, die durch seine Hamilton-Funktion ausgedrückt wird, ein Integral der Bewegungsgleichungen, sie bleibt demgemäß konstant.
>
> Die so entstandene ernste Schwierigkeit wird dadurch aufgelöst, daß wir die Teilchen der Materie als nur näherungsweise energetisch isolierte Komponenten betrachten können. Es besteht kein Zweifel, daß der Ausdruck für die wirkliche Energie des Systems auch Terme enthalten muß, die die Energie mehrerer Teilchen zugleich enthalten (Wechselwirkungspotentiale der Teilchen); diese Terme erlauben eine energetische Wechselwirkung zwischen den Teilchen (vom mathematischen Standpunkt aus verhin-

> dern sie die Aufspaltung des Gleichungssystems … in Systeme, die sich nur auf einzelne Teilchen beziehen). Da sich jedoch die Wechselwirkungskräfte zwischen Teilchen nur auf sehr kurze Entfernungen bemerkbar machen, sind diese gemischten Terme im Ausdruck für die Energie – die die Wechselwirkungspotentiale der Teilchen darstellen – , im weitaus größten Teil der Punkte des Phasenraumes vernachlässigbar klein gegenüber der kinetischen Energie der Teilchen oder gegenüber den Potentialen äußerer Felder. Insbesondere tragen sie äußerst wenig zur Berechnung verschiedener Mittelwerte bei; bei der Mehrzahl aller Rechnungen in der statistischen Mechanik können wir also solche Terme vernachlässigen, wir dürfen also in guter Näherung annehmen, daß die Energie des Systems gleich der Summe der Energien seiner Bestandteile ist; diese Bestandteile erscheinen also als Komponenten unseres Systems im oben definierten Sinne: Die vernachlässigten gemischten Terme sind hingegen vom prinzipiellen Standpunkt aus äußerst wichtig; gerade ihr Auftreten ermöglicht den Energieaustausch zwischen den Teilchen, auf dem die gesamte Methode der statistischen Mechanik beruht."

Of course statistical mechanics has not remained at that level but hat taken non-local interaction into consideration (Grosse, 1996).

Concerning the interpretation of statistical mechanics in terms of measurements Chintschin notes in the chapter on ergodicity (1964/52 f):

> "In den Darstellungen der statistischen Mechanik wird üblicherweise dieses Phasenmittel als theoretische Interpretation jeder physikalischen Größe betrachtet. Dabei werden aber entweder überhaupt keine Argumente für eine solche Wahl gegeben, oder man konstruiert eine spezielle Hypothese, die diese Wahl rechtfertigen soll; oft werden auch verschiedene Gründe zugunsten einer solchen Interpretation zitiert, wobei man zur gleichen Zeit sagt, daß diese Gründe nicht logisch zwingend sind, daß aber diese Interpretation allgemein angenommen sei, da die auf ihr basierende Theorie zu erfolgreichen Resultaten führe. *Diese letzte Methode scheint uns die wissenschaftlich korrekteste zu sein*." *Italics: MS.*

While physicists and engineers are talking about averages in a very naïve way mathematicians are far from having solved the problems. It is the same situation as with the variations. It has to be stated under what conditions the averages or the variations are to be taken, typically constant energy.

A special 'twig' of 'statistical' mechanics is the physics of soft matter and biological systems (Stark, 2007).

19.4.7 Discussion

Hamilton's equations are completely equivalent to the Lagrangean equation together with the degenerate kinematic state equation, their 'advantage' according to Lanczos being that the Hamiltonian energy function does not include time derivatives of the generalised location.

This argument is felt to be a pseudo-argument, overlooking that the explicit Euler-Lagrangean state description in terms of generalised extensity of motion and generalised location has exactly the same 'advantages' as the Hamiltonian state description with the additional advantages to be discussed. The 'argument' is due to the usual lack of an explicit concept of state, implying that any differential equation of order higher than 1 may be converted into a set of differential equations of order 1, most directly by introducing derivatives as states.

The Euler-Lagrangean equation has even more advantages than the Lagrangean and the Hamiltonian equations, both suffering from the same limitations to degenerate kinematics and to potential forces. The explicit Euler-Lagrangean equation for non-holonomic speed, the balance of aggregate momentum derived earlier has the advantage to be far more general and perfectly adapted to 'terrestrial' applications including aggregate momentum diffusion into the body and aggregate reversible storage and irreversible dissipation of aggregate momentum in the body under consideration and its environment.

Lanczos himself summarises in a 'box' amounting to wide and/or underlined print (1986/31):

> "Forces which are not derivable from a work function can still be characterized by the work done in a small displacement, but they do not yield to the general minimizing procedures of analytical mechanics."

He misses to note explicitly that these are exactly the problems of interest in engineering.

The present exposition clearly shows the complete equivalence with Newtonian mechanics developed as the balance of the quantity of motion ('*motus quantitas*': Bewegungsmenge), as an instance of the general balance of quantities. Already Lagrange has clearly stated that the minimum 'principle' is just a theorem valid under very 'narrow' conditions.

Keeping in mind the restrictive assumptions made in the derivations 'theoretical', 'analytical' or 'variational' mechanics are highly specialised aggregates of Newtonian mechanics, not in terms of the components of momentum and 'forces', but in terms of 'partial energies' and 'partial powers'.

Kane[TR] has stated the advantages of the procedure explicitly (1985/X):

> "Focussing attention on motions rather than configurations it [*Kane's method*] affords the analyst maximum physical insight. Not involving variations, such as those encountered in connection with virtual work, it can be presented at a relatively elementary mathematical level. Furthermore, it

enables one to deal directly with non-holonomic systems without having to introduce and subsequently to eliminate Lagrange multipliers." [*Addition*]*: MS.*

And in view of the inadequate teaching of mechanics mentioned earlier Kane[TR] continues:

"It follows that the resolution of the dilemma before us is to instruct students in the use of this method (which is often referred to as Kane's method). This book is intended as the basis for such instruction."

And the present exposition is hopefully providing additional insights and support.

In the introduction to his chapter on 'General Dynamics' (1962/99 ff) Synge notes (1962.C/100):

"General dynamical theory occupies a curious position in physics. Historically, it has been suggested by, and developed in terms of Newtonian dynamics of particles and rigid bodies. But we feel an urgent need to give it a wider scope, presenting it as a consistent mathematical theory *applicable to any physical system the behaviour of which can be expressed in Lagrangean or Hamiltonian form*. There is a temptation to present it as pure mathematics, and the exposition which follows is a compromise." *Italics: MS.*

All this is pretty incredible. As has been show the Lagrangean and the Hamiltonian theories are covering only a very small subset of classical aggregate mechanics. Not only in mechanics, systems 'the behaviour of which can be expressed in Lagrangean or Hamiltonian form' are in the minority. The situation, not the mathematics, is 'similar' to the consideration of linear systems instead of non-linear systems, *i. e.* a singular model in a multidimensional space of possible models.

So the title of Synge's chapter is grossly misleading in this respect. the same holds concerning 'any physical system'. The abstract meta-physics discussed in detail, the state space model is much more general, covering an even larger manifold of systems' behaviours, and not only of physical systems, but 'any systems'.

EVALUATIONS/ASSESSMENTS

The case of holonomic speed is conveniently treated as an instance of the more general case of non-holonomic speed, differing from the latter formally only in the degenerate kinematic state equation.

This approach permits to reduce the discussion of the 'variational principles of mechanics' to a short section. As has been mentioned earlier Lanczos' exposition of the latter suffers from serious defects in the conception of classical mechanics.

CONCLUSIONS

Although the literature abounds with examples of applications these are not subject of the present exercise devoted to elucidation of the foundations of classical mechanics.

19.5 Stability criteria

PROBLEMS

Stability considerations are clearly distinct from equilibrium considerations discussed so far.

MODELS

The general models have been outlined in the context of the elementary integrals of Cauchy's universal local balance of momentum.

GOALS/PLANS

At this stage of the manuscript the goals of this section are very limited. In detail only the criterion of least curvature will be treated and as example the stability of Kármán vortex streets.

The section will be concluded by some general considerations closely related to the problems of statistical mechanics mentioned by Chintschin.

19.5.1 Principle of least curvature

The invariant curvature of the path is defined as

$$1 / r^2 = d_t \mathbf{v}_u \, d_t \mathbf{v}_u / (\mathbf{v}_v \mathbf{v}_v)^2 .$$

According to the dynamic state equation

$$d_t \mathbf{v}_u = \mathbf{D}_u (\mathbf{s}, \mathbf{v})$$

it is vanishing along the whole path if the generalised 'forces' balance each other or are absent

$$\mathbf{D}_u (\mathbf{s}, \mathbf{v}) = \mathbf{0}_u .$$

Thus at steady state it is not only the 'straightest' but a straight path in configuration space

$$d_t \mathbf{s}_u = \mathbf{v}_u = \text{const} .$$

The short discussion of Hertz' principle by Sommerfeld is felt to be complete inadequate (1955/203-205).

19.5.2 Example: Kármán vortex street

> "Ich habe aber auch den Gegenstand vierzig Jahre mit mir herumgetragen, sodaß er denn freilich Zeit hatte, sich von allem Ungehörigen zu läutern."
>
> *Johann Wolfgang Goethe: 10.11.1823 (Eckermann, 1911/52).*

19.5.2.1 MOTIVATION

In the theory of resistance and propulsion vortex streets, the 'roller bearings' of convective momentum flows, play an essential role. Among the vortex streets, following laws of their own, stable Kármán vortex streets are the most widely known.

The study of Kármán vortex streets, their stability in particular, has given rise to a vast, highly specialised literature of its own (Lamb, 1932/224-29; Kotschin, 1954/186-215; Dolaptschiew, 1937, 1938), reviews on the subject to be found in the 'Advances in Applied Mechanics'.

Since fifty years, since dealing with vortex streets in the context of pulse jet propulsion the author, in view of the fundamental phenomenon, has been convinced that it must be possible to derive the stability criterion of Kármán vortex streets directly from 'some' general principle instead of going through some specialised mathematics, which happened to be developed right before von Kármán came across a phenomenon to which it could be applied (*Reference lost*).

In a recent paper on 'Explorieren – Entdecken – Testen' the role of convictions has been elaborated on among other 'practical' aspects guiding research (Steinle, 2008). The thesis underlying the work of the author and others at the MPIWG at Berlin already quoted at length (2002/100):

> "Scientists, while being well aware of the local specificities of their particular experiences, are at the same time convinced that there is something to be learned beyond the particular case, and that conviction plays an essential role in guiding research."

In the following Hertz' principle of least curvature will be applied to derive a stability criterion for Kármán vortex streets. These are conveniently studied in the observation space moving with the permanent configuration of the vortices, moving with the group or energy velocity relative to the fluid at large being at rest.

In this case the irrotational velocity field is steady and consequently the constraint becomes

$$Z = \int_m \partial_i v^2 \partial_i v^2 \, dm = \min .$$

To the knowledge of the author this most straightforward stability criterion has not yet been investigated.

19.5.2.2 STABILITY CRITERION

Kármán vortex streets, jets bounded by line vortices, provide examples of constrained systems with only two degrees of freedom in the position space. In the observation space moving with the group velocity the velocity field is steady and has as generalised 'coordinates' the distance λ between the vortices and their strengths Γ.

Normalising the generalised coordinates with the width h of the vortex street, the density ρ of the fluid, assumed to be constant, and the group energy E_G, results in the normalised distance

$$\alpha \equiv \lambda / (\pi h)$$

and the normalised vorticity

$$\phi \equiv (\Gamma / \lambda)^2 \rho h \lambda / E_G .$$

respectively. In terms of these normalised components the equilibrium positions of the vortex streets are

$$\phi = 1 / [\alpha/2 + \mathrm{tgh}(1 / \alpha)]$$

(Kotschin, 1954/213).

Hertz' principle of least curvature requires minimum curvature

$$\kappa = d^2\phi/d\alpha^2 / (1 + (d\phi/d\alpha)^2)^{3/2}$$

for stable conditions. The formal and finally numerical evaluation provides as result the normalised distance

$$\alpha_{\mathrm{Hertz}} = 1.1464$$

corresponding to the ratio

$$(\lambda / h)_{\mathrm{Hertz}} = 3.6015 .$$

As has been shown, in the range of interest the approximation

$$\kappa \approx d^2\phi/d\alpha^2$$

'looks' sufficient, but the location of the minimum is 2 % higher.

The values determined according to Hertz' criterion are 1 % higher than the values of the normalised distance

$$\alpha_{\mathrm{Kármán}} = 1.1344$$

and the corresponding ratio

$$(\lambda / h)_{\text{Kármán}} = 3.5639 ,$$

respectively, resulting from von Kármán's classical criterion

$$\cosh (1/\alpha_{\text{Kármán}}) = \sqrt{2}$$

also stated in the equivalent form

$$\sinh (1/\alpha_{\text{Kármán}}) = 1$$

(Sommerfeld, 1957/213-219).

The details of the numerical analysis in the Mathcad 8.3 environment are to be found on the website of the author. A 'disadvantage' of Hertz' stability criterion is that it does not result in a simple formula but requires numerical analysis. An attempt to derive von Kármán's criterion as an approximate result failed.

19.5.2.3 APPRAISAL

In view of the very intricate deductions in the literature no attempted has been made to trace the reasons for the very small difference, negligible in terms of curvature.

Even more difficult than judging the implications of the 'usual' derivations is the appraisal of the results of investigations including second order terms, part of a doctoral thesis worked out still under the guidance of Prandtl (Dolaptschiew 1937, 1938).

Similar stability considerations for jets bounded by vortex rings have been confirmed by data taken from a paper by Wille[R] (1952). The problem is that in this case the group velocity depends on the diameter of the vortex core, hard to be identified reliably from the photographs published, maybe so in general.

19.5.3 General criteria?

In view of the forgoing detailed example and the remarks by Chintschin concerning the conceptual problems faced in statistical mechanics the question arises how a meta-theory of stability and general stability criteria can be conceived.

The author having attended and followed a number of 'Symposia on Stability of Ships and Platforms' always found an adequate conceptual framework still missing. Today's problem is no longer static stability in calm water, solved by Archimedes, Euler and others and today a matter of careful bookkeeping, but the stability of ships and platforms in seaways, 'preferably' extreme.

Among the many conceptual problems encountered only few will be addressed. In general the systems under investigation are non-linear in 'stochastic' environments. Thus not individual trajectories are of interest but stochastic processes, all trajectories belonging to the ensemble under investigation.

The treatment of such notoriously difficult problems dates back to the second world war, to the solution of weapon guidance. A series of fifteen lectures by Wiener provides an early survey of theoretical approaches and their applications in many fields, two lectures devoted to a tentative 'A new approach to statistical mechanics' (Wiener, 1966).

Since the lectures of Wiener computers have become a common tool and with their help the intrinsic behaviour of non-linear systems has been studied in great detail. Non-linear systems starting from some initial conditions perform chaotic motions around strange attractors or may be asymptotically stable. And the question is how the 'basins of attraction' look like.

In studies of 'simple' systems fractal erosion of the basins of attraction with the change of system parameters has been observed. For ready reference the conclusions of a study are quoted (Soliman, 1995/739):

> "In this paper we have considered how the global response of a forced coupled non-linear system which has the ability to escape from a two-dimensional potential well may be examined and assessed. The phenomenon of fractal basin erosion, as observed in single-degree-of-freedom systems, was identified. By taking various cross-sections of the four-dimensional phase-space, we have described and quantified how basins of attraction evolve as system parameters are varied. Under increasing excitation, basin boundaries can become highly intertwined, in which they may have a non-integer fractal dimension. Although the fractal zone may initially be confined to a small region of phase-space, a relatively, small parameter change may result in rapid erosion and stratification of the whole basin. Consequently, since this may result in a substantial region of phase space having an infinitely textured homoclinic structure, due to the inherent uncertainties in the specification of the initial conditions, experienced by all dynamical systems to some degree both short- and long-term predictability can be lost. From a practical point of view this sudden fractal erosion, resulting in chaotic transient behaviour, may induce a loss of global transient stability of the system."

In view of this intricate behaviour under grossly simplified conditions no 'simple' general stability criteria are to be expected soon.

EVALUATIONS/ASSESSMENTS/CONCLUSIONS

The problems of stability and their solutions have only been scratched. The purpose was to show that the problems are far from being solved.

CLOSURE OF CHAPTER

EVALUATIONS/ASSESSMENTS

The reconstruction of 'analytical mechanics' in terms of the partial energy balances is extremely transparent and provides a wealth of insights into the essentials usually completely obscured by traditional formalistic expositions.

The clear distinction of speed and position spaces, the introduction of non-holonomic speeds is sufficient to arrive at the explicit Euler-Lagrangean equation, a model adequate to deal with systems of non-spherical aggregate inertia and/or including rotational motions in bodies and their environments.

Apart of the energetically neutral gyroscopic 'force' the aggregate 'forces', the partial powers, considered include aggregate diffusion, irreversible dissipation, reversible storage and production of aggregate momentum, partial energy.

CONCLUSIONS

The short exposition will hopefully be sufficient incentive soon to rewrite (not only) the pertinent sections in textbooks and lecture notes. It is felt to be irresponsible not (immediately) to stop the incredible waste of time, of resources and of motivation of students and teachers in adhering to eighteenth century derivations of equations not only hard to use but, even worse, of little use.

As an example rigid body dynamics will be discussed in detail, in a format adequate for practical application in realistic problems, typically bodies moving in fluids to be investigated in the subsequent chapter. In view of its importance rigid body kinematics will also be treated in detail, although kinematics in general is not the subject of the present treatise.